MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS,

FROM THE "EDUCATIONAL TIMES,"

WITH MANY

Papers and Solutions not published in the "Educational Times."

EDITED BY

W. J. C. MILLER, B.A.,

REGISTRAR

OF THE

GENERAL MEDICAL COUNCIL.

VOL. XL.

LONDON:

FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

1884.

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Mathematical Papers, &c.

Solved Questions.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

4641. (By the late Professor CLIFFORD, F.R.S.)—If a circular cubic with a double point O be cut by a circle in four points, A, B, C, D; and if OA, OB, OC, OD cut the circle again in E, F, G, H; show that any pair of straight lines joining these four points will be equally inclined to the bisectors of the angles between the tangents at O.

Solution by W. J. C. SHARP, M.A.

The inverse of the cubic about the node is a conic, the asymptotes of which are parallel to the nodal tangents, and the axes to the bisectors of the angles between these.

Now, if a, b, c, d, e, f, g, h be the points inverse to A, B, C ... H, the line joining ab will be parallel to that joining EF, since OA . OE = OB . OF and Oa . OA = Ob . OB, and therefore $\frac{\text{OE}}{\text{OF}} = \frac{\text{OB}}{\text{OA}} = \frac{\text{O}a}{\text{O}b}$.

Similarly all the connectors of a, b, c, d are parallel to the corresponding connectors of E, F, G, H. And each pair of lines connecting a, b, c, d are equally inclined to the axis of the conic. And therefore, &c.

[The property might be more generally enunciated, as it is true for the inverse of a conic, that is to say, for any bicircular quartic or circular cubic with a double point.]

7143. (By Professor SYLVESTER, F.R.S.)—If

$$\log F(x, y) = \Sigma \log \left(x - 2\cos\frac{2\lambda\pi}{\kappa}\, y\right),$$

where λ is to assume all values prime to κ and not exceeding $\frac{1}{2}(\kappa-1)$; prove that, when x, y are relative primes, $F(x, y)$ can have no *prime* factors other than divisors of κ or of the form $\kappa i \pm 1$.

Note by the PROPOSER.

If κ be any given number, $\phi\kappa$ the number of numbers λ less than κ and prime to it, and Fx the product of the $\phi\kappa$ factors $\left(x-\cos\frac{2\lambda\pi}{\kappa}\right)$, it is, in this Question, required to prove that, for all integer values of x, the factors of Fx prime to κ are of the form $\kappa i \pm 1$.

Ex. 1.—Let $\kappa = 8,\ \phi\kappa = 4,\ Fx = (x^2-2)^2$, and the odd factors of x^2-2 are of the form $8i \pm 1$.

Ex. 2.—Let $\kappa = 12,\ \phi\kappa = 4,\ Fx = (x^2-3)^2$, and the odd factors of x^2-3 not containing 3 are of the form $12i \pm 1$.

Ex. 3.—Let $\kappa = 18,\ \phi\kappa = 6,\ Fx = (x^3-3x+1)(x^3-3x-1)$, and the factors of $x^3-3x\pm1$ not containing 3 are of the form $18i\pm1$.

[Prof. SYLVESTER states that this last example is of paramount importance in his new theory concerning the resolution of integers into the sum or difference of two rational cubes. With the exception of the two numbers 66 and 74, all the numbers up to 100 inclusive can now either be resolved into two cubes or proved to be irresolvable, and it is likely that with further trial these two exceptions can be made to disappear.]

7227. (By the EDITOR.)—Show that (1) two sets of n things, whereof the individuals are marked 1, 2, 3, ... n, can be permuted so that no two individuals marked with the same number shall occupy the same place in each set, in $(n!)^2\left\{\frac{1}{2!}-\frac{1}{3!}+\ldots+\frac{(-1)^n}{n!}\right\}$ ways; and therefrom (2), if two examiners, working simultaneously, examine a class of 12 boys, the one in Classics and the other in Mathematics, so that the boys are examined individually, for 5 minutes each, in each subject, a suitable arrangement, such that no boy shall be wanted by both examiners at once, can be made in 84407190782745600 ways.

Solution by W. J. C. SHARP, M.A.; D. BIDDLE; *and others.*

1. The first set may be permuted in $n!$ ways; and with any particular arrangement, *ex. gr.*, 1, 2, 3, ... n, the following numbers of arrangements of the second set will be excluded by the conditions:—

$(n-1)!$	in which	1	stands	first;
$(n-1)!-(n-2)!$	,,	2	,,	2nd, and 1 is not first;
$(n-1)!-2(n-2)!+(n-3)!$	,,	3	,,	3rd, and 1 is not first;
&c.	&c.	&c.		or 2 second;

$$(n-1)!-(n-1)(n-2)!+\frac{(n-1)(n-2)}{1\,.\,2}(n-3)!-\&c.,$$

in which n is last, and no other in the same place as in the chosen arrange-

ment of the first set. These exclusions amount to

$$n(n-1)!-\frac{n(n-1)}{1.2}(n-2)!+\frac{n(n-1)(n-2)}{1.2.3}(n-3)!-\&c.$$

$$=n!\left\{1-\frac{1}{1.2}+\frac{1}{1.2.3}-\ldots+(-1)^{n-1}\frac{1}{n}\right\}$$

and therefore the number of included sets is

$$n!\left\{\frac{1}{1.2}-\frac{1}{1.2.3}+\&c. \quad +(-1)^n\frac{1}{n!}\right\}.$$

Hence the number of ways is that given in the question.

2. Applying the formula in (1) to the particular case, the number of ways is $(12!)^2\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\ldots+\frac{1}{12!}\right)$, which gives the stated result.

II. *Solution by the* Rev. T. P. Kirkman, M.A., F.R.S.

1. The solution of the general question is readily seen by the following discussion of the particular case thereof, which forms parts (2):—

2. If the number of the boys were three, the solution would be, giving to each examiner every possible roll-call,

231 312 123 213 132 321
123 231 312 132 321 213

312 123 231 321 213 132
123 231 312 132 321 213,

which is $S_3 . 3!$, $S_3 = 2$ being the number of the all-disturbed permutations of 123; $(3! = 3.2.1)$.

For 12 boys the solution is $S_{12} . (12)!$, S_{12} being the number of the all-disturbed permutations of 123 ... 9*abc*. Each permutation is distinguished by its *circles*. Thus 23164589*abc*7 is one of the number $B_{6,3,3}$, having one circle of six and two circles of three. S_{12} is the sum of such B_p.

In my memoir *On the Theory of Groups and many-valued Functions* (Manchester Memoirs, 1862), at p. 284, there is a demonstration that the number of all-disturbed arrangements of N elements for the partition

$$Aa+Bb+\ldots+Jj=N, \ (J>1), \ (p),$$

showing *a* circles of A, *b* circles of B, &c., $A>B$, $B>C$, &c., is

$$\frac{N!}{a!\,b!\ldots j!\,A^a B^b \ldots J^j}=B_p.$$

The number of skeleton groups there spoken of, is that of the different arrangements of N so partitioned that can be written under 12 ... 9*abc* ..., the unity of N elements.

The partitions *p* before us of N = 12 are 12·1, 10·1 + 2·1, 9·1 + 3·1, 8·1 + 2·2, &c.; say, (12), (10) 2, 93, 822, 84, 75, 732, 66, 642, 633, 6222, 543, 5322, 552, 444, 4422, 4332, 42222, 3333, 33222, 222222. The above formula gives all the B_p for N = 12, as follows:—

$$B_{12}=11!;\quad B_{10,2}=\frac{12!}{10.2};\quad B_{9,3}=\frac{12!}{9.3};\quad B_{8,22}=\frac{12!}{2!\,8.2^2};$$

$$B_{8,4}=\frac{12!}{8.4};\quad B_{7,5}=\frac{12!}{7.5};\quad B_{7,3,2}=\frac{12!}{7.3.2};\quad B_{6,6}=\frac{12!}{2!\,6^2};$$

$$B_{6,4,2} = \frac{12!}{6.4.2}; \quad B_{6,3,3} = \frac{12!}{2!\,6.3^2}; \quad B_{6,2,2,2} = \frac{12!}{3!\,6.2^3};$$
$$B_{5,4,3} = \frac{12!}{5.4.3}; \quad B_{5,3,2,2} = \frac{12!}{2!\,5.3.2^2}, B_{5,5,2} = \frac{12!}{2!\,5^2.2};$$
$$B_{4,4,4} = \frac{12!}{3!\,4^3}; \quad B_{4,4,2,2} = \frac{12!}{2!\,2!\,.4^2.2^2}; \quad B_{4,3,3,2} = \frac{12!}{2!\,4.3^2.2};$$
$$B_{4,2,2,2,2} = \frac{12!}{4!\,4.2^4}; \quad B_{3,3,3,3} = \frac{12!}{4!\,3^4};$$
$$B_{3,3,2,2,2} = \frac{12!}{2!\,3!\,3^2.2^3}; \quad B_{2,2,2,2,2,2} = \frac{12!}{6!\,2^6}.$$

We can find $S_1, S_2, \ldots S_N$ very easily from the formula $N! = (1+S)^N$, where for S^e we have to write S_e; $N > 1$ and $S_1 = 0$; or from

$$S_n - nS_{n-1} = (-1)^n, \; n > 1, \text{ and } S_{2-1} = 0.$$

This gives $S_{12} = 176214841$; thus the number required for N_{12} is $N!\,176214841$, which gives the stated result.

7268. (By W. S. M'Cay, M.A.)—Prove that two equal non-intersecting circles are polar reciprocals to an imaginary parabola.

Solution by Arthur Hill Curtis, LL.D., D.Sc.

Let the equation of one circle be $(x-a)^2+(y-b)^2-r^2=0$, ...(1), and that of the parabola be $y^2-px=0$ .... (2); then the equation of a tangent to the circle at the point (x', y') will be

$$(x-a)(x'-a)+(y-b)(y'-b)-r^2=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

and if (x_1, y_1) be the pole of this tangent with regard to the parabola, (3) must be identical with $2y_1y-p(x_1+x)=0$; hence the conditions

$$\frac{y'-b}{x'-a} = -\frac{2y_1}{p}, \quad \frac{a(x'-a)+b(y'-b)+r^2}{x'-a} = -x_1, \quad (x'-a)^2+(y'-b)^2-r^2=0;$$

from which equations, on eliminating $x'y'$, we obtain

$$\{2by_1 - p(a+x_1)\}^2 - r^2\{p^2+4y_1^2\} = 0,$$

or, transferring the origin of coordinates to the centre of the circle (1), we obtain, as the equation of the polar reciprocal required,

$$\{2b(y+b)-p(x+2a)\}^2 - r^2\{p^2+4(y+b)^2\} = 0 \quad \ldots\ldots\ldots (4).$$

Now, that this should be a circle, we must have $b=0$, and $p^2=-4r^2$, which reduces (4) to $(x+2a)^2+y^2-r^2=0$ .... (5); while (1) and (2) are transformed to

$$x^2+y^2-r^2=0 \text{ and } y^2-2r\sqrt{-1}\,(x+a)=0 \quad \ldots\ldots\ldots (6, 7).$$

Of these (5) and (6) represent two circles of radius r, which intersect, touch, or are wholly external to each other according as $a \lesseqgtr r$, while (7) represents an imaginary parabola, whose vertex only is real, and bisects the distance between the centres of the two circles.

7331. (By Professor MALET, F.R.S.)—If $\Delta \equiv 1-kx^2$, prove that

$$14\int_0^1 \Delta^{\frac{3}{2}} \log\Delta\, dx - 8\int_0^1 \frac{\log\Delta}{\Delta^{\frac{1}{2}}}\, dx = 9\int_0^1 \Delta^{\frac{3}{2}}\, dx + 3\,(1-k)^{\frac{3}{2}}\,[2\log(1-k)-3].$$

Solution by HANUMANTA RAN; Prof. NASH, M.A.; *and others.*

Since $\displaystyle\int \Delta^{\frac{3}{2}} \log\Delta\, dx = x\,\Delta^{\frac{3}{2}}\log\Delta + \int \left\{ \tfrac{3}{2}\cdot 2kx\,\frac{\log\Delta}{\Delta^{\frac{1}{2}}} + \frac{2kx}{\Delta^{\frac{1}{2}}} \right\} x\, dx$

$$= x\,\Delta^{\frac{3}{2}}\log\Delta + \tfrac{4}{3}\int (1-\Delta)\,\frac{\log\Delta}{\Delta^{\frac{1}{2}}}\, dx - \tfrac{2}{3}\int x d\,\Delta^{\frac{3}{2}},$$

we have, therefore, $\displaystyle 14\int \Delta^{\frac{3}{2}}\log\Delta\, dx - 8\int \frac{\log\Delta}{\Delta^{\frac{1}{2}}}\, dx$

$$= 6x\,\Delta^{\frac{3}{2}}\log\Delta - 9\int x\, d\,\Delta^{\frac{3}{2}} = 6x\,\Delta^{\frac{3}{2}}\log\Delta - 9x\,\Delta^{\frac{3}{2}} + 9\int \Delta^{\frac{3}{2}}\, dx.$$

Integrating between the limits 1 and 0, we obtain the required result.

5850. (By Professor SYLVESTER, F.R.S.)—1. Suppose an arborescence subject to the law that at every joint each stem or branch splits up into m, the main stem being reckoned as a free branch. Prove that, if n is the number of such joints, $(m-1)n+2$ will be the number of free branches.

2. If $m = 2$, *i.e.* for the case of dichotomous ramification, it will be found that, making as above no distinction between the main stem and any free branch, the number of *distinct forms of arborescence*, when there are 1, 2, 3, 4, 5, 6, 7, 8, 9, &c. joints, will be respectively 1, 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, &c. Let such number be called N. Required to express N generally in terms of n, when the arborescence is dichotomous.

Solution by W. J. C. SHARP, M.A.

The total number of branches including the stem is $mn+1$, and of these $n-1$ are not free; therefore, the number of free branches

$$= mn+1-(n-1) = n\,(m-1)+2.$$

Hence, if $m = 2$, the number of free branches is $n+2$.

If now it be assumed that all systems which have the same number of joints with two free branches (free joints, say) belong to the same species; then, denoting by p the number of free joints, and by q the number of joints with one free branch, $2p+q = n+2$ the number of free branches. Also q cannot exceed $n-2$.

Then N is the number of integer solutions of the above equation which satisfies the condition. If $n = 1$, the equation, is not applicable as there is one joint with three free branches.

For $n = 2, 3, 4, 5, 6, 7$, &c., the values of N are 1, 1, 2, 2, 3, 3, &c., N being $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$ according as n is even or odd.

7344. (By T. Muir, M.A., F.R.S.E.)—Prove the theorem of continuants which for the case of the 4th order is

$$\mathrm{K}(a-1, a, a, a+1) = a\mathrm{K}(a, a, a).$$

Solution by the Proposer.

Making use twice of the identity

$$\mathrm{K}(a+\omega, b, c, d, \ldots) = \mathrm{K}(a, b, c, d, \ldots) + \omega\mathrm{K}(b, c, d, \ldots),$$

we have $\mathrm{K}(a-1, a, a, a+1) = \mathrm{K}(a, a, a, a+1) - \mathrm{K}(a, a, a+1)$

$$= \mathrm{K}(a, a, a, a+1) - \mathrm{K}(a, a, a) - \mathrm{K}(a, a).$$

But the first term of this expansion equals $(a+1)\mathrm{K}(a, a, a) + \mathrm{K}(a, a)$. Hence the theorem is established. As an easy deduction from this, we have

$$\frac{a}{a+1+}\;\frac{1}{a+}\;\frac{1}{a+}\;\frac{1}{a-1}+\frac{1}{a+}\;\frac{1}{a+}\;\frac{1}{a} = 1.$$

7334. (By C. E. McVicker, M.A.)—Adopting the usual notation for the radii of the circles connected with a plane triangle, prove that

$$r_1^3+r_2^3+r_3^3-r^3 = 64\mathrm{R}^3 - 6\mathrm{R}(a^2+b^2+c^2) \quad\ldots\ldots\ldots\ldots\ldots(1);$$

$$32\mathrm{R}^3 - 6\mathrm{R}(r_1^2+r_2^2+r_3^2+r_2) + (r_1^3+r_2^3+r_3^3-r^3) = 0 \quad\ldots\ldots\ldots(2).$$

Solution by Hanumanta Rau; G. Heppel, M.A.; *and others.*

1. Let $r_1+r_2+r_3 = u, \quad r_1r_2+r_2r_3+r_3r_1 = v, \quad r_1r_2r_3 = w,$

then $u = 4\mathrm{R}+r, \; v = s^2, \; w = rs^2$, and $4\mathrm{R}r+r^2-s^2 = -\frac{1}{2}(a^2+b^2+c^2)$.

Now $r_1^3+r_2^3+r_3^3 = u^3-3uv+3w = 64\mathrm{R}^3+48\mathrm{R}^2r+12\mathrm{R}r^2+r^3-12\mathrm{R}s^2$,

$\therefore\; r_1^3+r_2^3+r_3^3-r^3 = 64\mathrm{R}^3+12\mathrm{R}(4\mathrm{R}r+r^2-s^2) = 64\mathrm{R}^3-6\mathrm{R}(a^2+b^2+c^2)$.

2. Again $r_1^2+r_2^2+r_3^2 = u^2-2v = 16\mathrm{R}^2+8\mathrm{R}r+r^2-2s^2$,

$\therefore\; r_1^2+r_2^2+r_3^2+r^2 = 16\mathrm{R}^2+2.(4\mathrm{R}r+r^2-s^2) = 16\mathrm{R}^2-(a^2+b^2+c^2)$,

therefore $32\mathrm{R}^3-6\mathrm{R}(r_1^2+r_2^2+r_3^2+r^2)+r_1^3+r_2^3+r_3^3-r^3 = 0$.

7341. (By A. Martin, B.A.)—Solve the equations

$$yz(y+z-x) = a, \quad zx(z+x-y) = b, \quad xy(x+y-z) = c.$$

Solution by W. W. Taylor, M.A.; Hanumanta Rau; *and others.*

If we put $p = x+y+z, \quad q = xy+yz+zx, \quad r = xyz,$

the given equations may be written

$$yzp-2r = a, \quad zxp-2r = b, \quad xyp-2r = e \quad\ldots\ldots\ldots(1, 2, 3).$$

By addition, and from (1) and (2), we obtain, respectively,

$$qp-6r = a+b+c, \quad zp^2r = (a+2r)(b+2r) \ldots\ldots\ldots \ldots (4, 5).$$

By adding (5) and two similar equations, and from (1), (2), (3), we have

$$p^3r = (ab+bc+ca)+4(a+b+c)r+12r^2 \ldots\ldots\ldots\ldots\ldots (6),$$

$$(a+2r)(b+2r)(c+2r) = x^2y^2z^2p^3 = p^3r^2 = r\left[ab+bc+ca+4(a+b+c)r+12r^2\right].$$

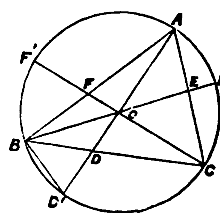
On simplification, this equation becomes $4r^3-(ab+bc+ca)r = abc$, a cubic in r; and when r is found, p can be found from (6), and q from (4).

The values of x, y, z are then the roots of the cubic $x^3-px^2+qx-r=0$.

7335. (By W. J. C. Sharp, M.A.)—If O be the centre of the circle drawn round the triangle ABC, and AO, BO, CO be produced to meet the opposite sides in D, E, F, and the circle in D′, E′, F′ respectively; prove that

$$\frac{DD'}{AD}+\frac{EE'}{BE}+\frac{FF'}{CF} = 1, \quad BD.DC+OD^2 = AE.EC+OE^2 = AF.FB+OF^2.$$

Solution by Hanumanta Rau; R. Knowles, B.A.; *and others.*



$$AD = 2R.\frac{\sin B \sin C}{\cos(B-C)}; \quad DD' = 2R.\frac{\cos B \cos C}{\cos(B-C)};$$

therefore $\dfrac{DD'}{AD} = \cot B \cot C$;

therefore, since $A+B+C = 180°$,

$$\frac{DD'}{AD}+\frac{EE'}{BE}+\frac{FF'}{CF} = 1;$$

the second part of the question follows at once from Euclid III. 35.

7349. (By Sarah Marks.)—If a number consist of 7 digits, whose sum is 59, show that the probability that it will be exactly divisible by 11 is $\frac{4}{21}$.

Solution by Professor Roy; G. Heppel, M.A.; *and others.*

In order that the number may be divisible by 11, the digits in the even places must together be equal to 24.

Now set	8888999	gives	4	such arrangements	out of	35
	7889999	,,	24	,,	,,	105
	7799999	,,	0	,,	,,	21
	6899999	,,	12	,,	,,	42
	5999999	,,	0	,,	,,	7,

so that there are 40 suitable arrangements out of 210, and the chance is $\frac{4}{21}$.

[illegible]. (By the Rev. [illegible], M.A.) — If the [illegible] of two circles $A = 0$, $B = 0$ are concentric with the [illegible] of the intersections of $C = 0$, $D = 0$, then [illegible]; and if this property hold for the pairs A, B; C, D, and also for the pairs A, C; B, D, prove that it will likewise hold for the pairs A, D; B, C.

Solution by W. J. C. MILLER, B.A.

If a, b and a', b' be two points, the antipoints are

[illegible]

and [illegible],

so that the points and antipoints are concentric, and the first proposition follows at once, since the common chords bisect each other.

Again, if a, b, c, d be the centres of the circles, then, if the property hold for A, B; C, D, the perpendiculars from a and b upon $A-B=0$, and those from c and d upon $C-D=0$, meet these lines at their intersection; and if the property hold for A, C; B, D, the perpendiculars from a and c upon $A-C=0$ and from b and d upon $B-D=0$ meet these lines at the same point, and these conditions involve the identity of the four lines $A-B=0$, $A-C=0$, $C-D=0$, and $B-D=0$, which are, therefore, also identical with $A-D=0$ and $B-C=0$, which proves the second proposition, the circles A, B, C, D forming a system through two points.

[If the intersections be imaginary, the antipoints are real, and *vice versâ*.]

7154. (By the Rev. G. RICHARDSON, M.A.) — If three circles whose centres are O_1, O_2, O_3 and radii r_1, r_2, r_3 respectively, be coaxal, prove that

$$r_1^2 \cdot O_2O_3 + r_2^2 \cdot O_3O_1 + r_3^2 \cdot O_1O_2 + O_2O_3 \cdot O_3O_1 \cdot O_1O_2 = 0.$$

Solution by J. P. JOHNSTON, B.A.; J. O'REGAN; *and others.*

Take the line on which the centres lie and the radical axis as axes of coordinates, then the equations of the circles are

$$x^2+y^2-2k_1x+\delta^2=0,\quad x^2+y^2-2k_2x+\delta^2=0,\quad x^2+y^2-2k_3x+\delta^2=0,$$

where k_1, k_2, k_3 are the distances of centres from the radical axes. Then

$$r_1^2=k_1^2-\delta^2,\quad r_2^2=k_2^2-\delta^2,\quad r_3^2=k_3^2-\delta^2;$$

therefore

$$r_1^2(k_2-k_3)+r_2^2(k_3-k_1)+r_3^2(k_1-k_2)=k_1^2(k_2-k_3)+k_2^2(k_3-k_1)+k_3^2(k_1-k_2)$$
$$=-(k_2-k_3)(k_3-k_1)(k_1-k_2).$$

But $k_2-k_3=O_2O_3$, &c.; therefore the stated result follows.

7346. (By D. EDWARDES.)—Prove that, whatever be the value of n,

$$\int_0^{\frac12\pi}\int_0^{\frac12\pi}(1-\sin\theta\cos\phi)^{\frac12 n}\sin\theta\,d\theta\,d\phi=\frac{\pi}{n+2}.$$

Solution by Dr. CURTIS; BELLE EASTON; *and others.*

The limits of integration show that this integral is extended to the surface of a quadrantal triangle on the surface of a sphere of radius unity, or, taking as axes of coordinates x, y, z, the radii of the sphere drawn to the three angular points of the triangle, and supposing the radius vector to any assumed point within the area to make with x, y, z, angles α, β, γ, we have $\alpha=\theta$, $\cos\beta=\sin\alpha\cos\phi$, $\cos\gamma=\sin\alpha\sin\phi$; therefore, if $d\Omega$ denote the element of the area, we have

$$\int_0^{\frac12\pi}\int_0^{\pi}(1-\sin\theta\cos\phi)^{n}\sin\theta\,d\theta\,d\phi=\iint(1-\cos\beta)^{\frac12 n}\,d\Omega$$

$$=\int_0^{\frac12\pi}\int_0^{\frac12\pi}2^{\frac12 n}\sin^n\tfrac12\beta\sin\beta\,d\beta\,d\psi=2^{\frac12 n+1}\int_0^{\frac12\pi}\int_0^{\frac12\pi}\sin^{n+1}\tfrac12\beta\cos\tfrac12\beta\,d\beta\,d\psi$$

$$=2^{\frac12 n}\pi\int_0^{\frac12\pi}\sin^{n+1}\tfrac12\beta\cos\tfrac12\beta\,d\beta=\frac{2^{\frac12 n+1}\pi}{n+2}\left(_0^{\frac12\pi}\sin^{n+2}\tfrac12\beta\right)=\frac{\pi}{n+2}\,\frac{2^{\frac12 n+1}}{2^{\frac12(n+2)}}=\frac{\pi}{n+2}.$$

7192. (By Professor MATZ, M.A.)—Show that the sum of the series for

$$I=\int_0^{\frac12\pi}\frac{\sin\theta}{\theta}\,d\theta \text{ is } 1\cdot3749833960=\frac{11}{8}\text{ nearly.}$$

Solution by W. H. BLYTHE, M.A.; Professor NASH, M.A.; *and others.*

The series required is

$$a-\frac{a^3}{3\,.\,3\,!}+\frac{a^5}{5\,.\,5\,!}-\frac{a^7}{7\,.\,7\,!}+\text{\&c., where } a=\frac{\pi}{2};$$

and the annexed computation shows the result to be that stated in the Question.

$\log a$	$=\log\pi-\log 2$	$\log\frac{1}{5\,!}$	$=\bar{3}\cdot9208187$
	$=\cdot1961199$		
$\log a^3$	$=\cdot5783597$		
$\log a^5$	$=\cdot9605995$	$\log\frac{1}{7\,!}$	$=\bar{4}\cdot2975694$
$\log a^7$	$=1\cdot3428383$		
$\log a^9$	$=1\cdot7250781$	$\log\frac{1}{9\,!}$	$=\bar{6}\cdot2402369$
$\log a^{11}$	$=2\cdot1073179$		
$\log a^{13}$	$=2\cdot4895577$	$\log\frac{1}{11\,!}$	$=\bar{8}\cdot1988442$
$\log a^{15}$	$=2\cdot8717975$		
$\log\frac{1}{3\,!}$	$=\bar{1}\cdot2218487$	$\log\frac{1}{13\,!}$	$=\overline{10}\cdot0057296$
		$\log\frac{1}{15\,!}$	$=\overline{13}\cdot6835103,$

$$\log(\text{1st term}) = \log a = \cdot 1961199$$

$$\log(\text{2nd term}) = \log a^3 + \log\frac{1}{3\,!} - \log\ 3 = \bar{1}\cdot 3230871$$

$$\log(\text{3rd term}) = \log a^5 + \log\frac{1}{5\,!} - \log\ 5 = \bar{2}\cdot 1824482$$

$$\log(\text{4th term}) = \log a^7 + \log\frac{1}{7\,!} - \log\ 7 = \bar{4}\cdot 7953097$$

$$\log(\text{5th term}) = \log a^9 + \log\frac{1}{9\,!} - \log\ 9 = \bar{5}\cdot 0110725$$

$$\log(\text{6th term}) = \log a^{11} + \log\frac{1}{11\,!} - \log 11 = \bar{7}\cdot 2647694$$

$$\log(\text{7th term}) = \log a^{13} + \log\frac{1}{13\,!} - \log 13 = \bar{9}\cdot 3813439$$

$$\log(\text{8th term}) = \log a^{15} + \log\frac{1}{15\,!} - \log 15 = \overline{11}\cdot 3792165.$$

[The series is evidently convergent, for each alternate is positive and negative, and each successive term causes the sum to be either greater or less than some fixed limit, the terms also diminishing by a ratio less than unity.]

1st term	=	1·5707963268
2nd ,,	=	·2104200000
difference	=	1·3603763268
3rd term	=	·0152211724
sum	=	1·3755974992
4th term	=	·0006241800
diff.	=	1·3749733192
5th term	=	·0000102583
sum	=	1·3749835775
6th term	=	·0000001839
diff.	=	1·3749833936
7th term	=	·0000000024
sum	=	1·3749833960
8th term	=	·0000000000
Result	=	1·3749833960.

7317. (By Asutosh Mukhopâdhyây.) — Any number (m) of tangents are drawn to a parabola, such that the arcs between the points of contact subtend equal angles at the focus. If $2a$ be the angle which the axis of the parabola makes with the radius vector drawn to the adjacent point of contact, prove that the product of the perpendiculars n the focus on the tangents varies inversely as sin ma.

Solution by A. H. Curtis, LL.D, D.Sc.

The solution of this question may be obtained by reciprocation from a particular case of a theorem (an extension of Cotes' theorem), due to the late Prof. MacCullagh, which, so far as it concerns the present question, may be enunciated as follows:—If any point O, taken on the circumference of a circle of radius a, be joined to all the angular points of a regular polygon of m sides inscribed in the circle, and the angle subtended at the centre by the point O and any angular point of the polygon, *e.g.* the adjacent one, be denoted by 2α, the equation whose roots are the squares of these joining lines, $r_1, r_2, \ldots r_m$, is of the form

$$z^m - Az^{m-1} + Bz^{m-2} - \&c. \quad + (-1)^m 4a^{2m} \sin^2 m\alpha = 0,$$

where the coefficients A, B, &c., are functions of a, and therefore

$$(r_1 \cdot r_2 \ldots r\ \ )^2 = 4a^{2m} \sin^2 m\alpha, \text{ or } r_1 \cdot r_2 \ldots r_m = 2a^m \sin m\alpha.$$

If, taking O as origin, we reciprocate this theorem, we obtain the one proposed; for 2α is double of the angle between the diameter through O (the axis of the parabola), and a perpendicular to r_1 (co-directional with the *adjacent* tangent to the parabola into which the circle reciprocates), and is consequently identical with the angle expressed by 2α in the question.

The proof of MacCullagh's theorem adapted to the special case is as follows:—Let r_k denote the line joining O to the k^{th} angular point, then

$$z = r_k^2 = 4a^2 \sin^2 \tfrac{1}{2}\left(2\alpha + \frac{k2\pi}{m}\right),$$

or $$\frac{z}{2a^2} = 2\sin^2 \tfrac{1}{2}\left(2\alpha + \frac{k2\pi}{m}\right) = 1 - \cos\left(2\alpha + \frac{k2\pi}{m}\right),$$

therefore $$\cos\left(2\alpha + \frac{k2\pi}{m}\right) = \left(1 - \frac{z}{2a^2}\right).$$

Now, if $\cos\phi = u$, $2\cos m\phi = (2u)^m - m(2u)^{m-2} + \dfrac{m \cdot m-3}{1 \cdot 2}(2u)^{m-4} - \&c.$,

therefore $$2\cos(2m\alpha) = 2\cos(2m\alpha + 2k\pi) = 2\cos m\left(2\alpha + \frac{k2\pi}{m}\right)$$

$$= \left\{2\left(1 - \frac{z}{2a^2}\right)\right\}^m - m\left\{2\left(1 - \frac{z}{2a^2}\right)\right\}^{m-2} + \frac{m \cdot m-3}{1 \cdot 2}\left\{2\left(1 - \frac{z}{2a^2}\right)\right\}^{m-4}$$

— &c. (Murphy's *Theory of Equations*, pp. 32 and 75);

the term independent of z in this equation, which will result on putting

$z = 0$, is $$2^m - m2^{m-2} + \frac{m \cdot m-3}{1 \cdot 2}2^{m-4} - \&c. \ -2\cos(2m\alpha),$$

or $2\cos m(\cos^{-1} 1) - 2\cos(2m\alpha)$, or $2(1 - \cos 2m\alpha)$, or $4\sin^2 m\alpha$,

therefore $$\frac{r_1^2 \cdot r_2^2 \ldots r_m^2}{(a^2)^m} = 4\sin^2 m\alpha,$$

and therefore $$r_1^2 \cdot r_2^2 \ldots r_m^2 = 2a^m \sin m\alpha.$$

7189. (By Professor Sylvester, F.R.S.)—Sum the series

$$1+(x-i)+\frac{(x-2i)(x-2i-1)}{1\,.\,2}+\frac{(x-3i)(x-3i-1)(x-3i-2)}{1\,.\,2\,.\,3}+\ldots\ldots$$

Solution by the Proposer.

The answer may be got by calling the sum u_x, and satisfying the equations $u_x-u_{x-1}-u_{x-i-1}=0,\quad u_0=1,\quad u_1=1,\quad u_2=1,\ldots u_{i-1}=1.$
Thus *ex. gr.*, we have

$$1+(x-1)+\frac{(x-2)(x-3)}{1\,.\,2}+\frac{(x-3)(x-4)(x-5)}{1\,.\,2\,.\,3}$$
$$+\ldots=\frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^x-\left(\frac{1-\sqrt{5}}{2}\right)^x\right\}.$$

6938. (By C. Morgan, B.A.)—If ABCDEF be a rectilineal figure, prove that the sum of the tangents of its interior angles is equal to the difference between the sums of the products of the tangents taken 3 and 5 together.

Solution by the Proposer; Professor Nash, M.A.; *and others.*

$$\tan\left[(A+B)+(C+D)+(E+F)\right]=$$
$$\frac{\tan(A+B)+\tan(C+D)+\tan(E+F)-\tan(A+B)\,.\tan(C+D)\,.\tan(E+F)}{\text{some denominator.}}$$

But $A+B+C+D+E+F=4\pi$; therefore the numerator of the above fraction is zero; or, putting $a, b, c\ldots$ for $\tan A$, $\tan B\ldots$, we have

$$\frac{a+b}{1-ab}+\frac{c+d}{1-cd}+\frac{e+f}{1-ef}-\frac{(a+b)(c+d)(e+f)}{(1-ab)(-1cd)(1-ef)}=0,$$

or $$a-acd-aef+acdef+b-bcd-bef+bcdef+c-cab-cef+cabef$$
$$+d-dab-def+dabef+e-eab-ecd+eabcd+f-fab-fcd+fabcd$$
$$-ace-bce-ade-bde-acf-bcf-adf-bdf=0,$$

or $$a+b+c+d+e+f=\left(\begin{matrix}acd+aef+\ldots\\ \text{20 terms}\end{matrix}\right)-\left(\begin{matrix}acdef+bcdef+\ldots\\ \text{6 terms}\end{matrix}\right).$$

7376. (By Professor Cayley, F.R.S.)—Show how the construction of a regular heptagon may be made to depend on the trisection of the angle $\cos^{-1}\left(\frac{1}{2\sqrt{7}}\right)$.

Solution by R. RAWSON; Professor MATZ, M.A.; *and others.*

Putting θ for the angle subtended by the side of the heptagon, we have $7\theta = 2\pi$, and $\sin 7\theta = \sin\theta\,(64\cos^6\theta - 80\cos^4\theta + 24\cos^2\theta - 1) = 0$;
hence, if $4\cos^2\theta = z$, we have $z^3 - 5z^2 + 6z - 1 = 0$ .... (1);
and, by the relation $z = v + \frac{5}{3}$, equation (1) is transformed into

$$v^3 - \tfrac{7}{3}v - \tfrac{7}{27} = 0 \qquad (2).$$

The trigonometrical solution of (2) is well known to be $v = \frac{2}{3}\sqrt{7}\cos\alpha$,

where $$\cos 3\alpha = \frac{7}{2\,.\,27}\left(\frac{9}{7}\right) = \frac{1}{2\sqrt{7}};$$

when therefore 3α, or $\cos^{-1}\dfrac{1}{2\sqrt{7}}$, is trisected,

then $4\cos^2\theta = 2\cos 2\theta + 2 = \frac{1}{3}(5 + 2\sqrt{7}\cos\alpha)$ is readily constructed.

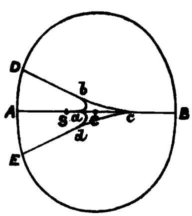
[It may be remarked that for $z = 2\cos\theta + \frac{1}{3}$, we have also the equation $z^3 - \frac{7}{3}z - \frac{7}{27} = 0$, which is identical with (2) above.]

7379. (By Professor WOLSTENHOLME, M.A.) — In the limaçon $r = a + b\cos\theta$, if $a > 2b$, prove that the length of the whole arc of the evolute, and the whole area of the evolute, will be, respectively,

$$4\left\{\frac{a(a^2 - 3b^2)}{a^2 - 4b^2} - (a^2 - b^2)^{\frac{1}{2}}\right\}; \quad \frac{\pi}{9}\left\{\frac{(a^2 - b^2)^{\frac{3}{2}}}{(a^2 - 4b^2)^{\frac{1}{2}}} - a^2 - \tfrac{1}{2}b^2\right\}.$$

Solution by D. EDWARDES; BELLE EASTON; *and others.*

We have $$\rho = \frac{(a^2 + b^2 + 2ab\cos\theta)^{\frac{3}{2}}}{a^2 + 2b^2 + 3ab\cos\theta}.$$



When $\cos\theta = -\dfrac{b}{a}$, there is a cusp on the evolute, and at this point $\rho = (a^2 - b^2)^{\frac{1}{2}}$. The radii of curvature at the vertices are $\dfrac{(a+b)^2}{a+2b}$ and $\dfrac{(a-b)^2}{a-2b}$.

Hence evidently the length

$$= 2\left[\frac{(a+b)^2}{a+2b} + \frac{(a-b)^2}{a-2b} - 2(a^2 - b^2)^{\frac{1}{2}}\right] = 4\left[\frac{a(a^2 - 3b^2)}{a^2 - 4b^2} - (a^2 - b^2)^{\frac{1}{2}}\right].$$

Let E be the area of the evolute and A that of the limaçon, and let s be the arc of the limaçon measured from the farther apse. Then $E + A = \int \rho\, ds$, between the limits 0 and π of θ, that is, $E + A = \displaystyle\int_0^\pi \frac{(a^2 + b^2 + 2ab\cos\theta)^2}{a^2 + 2b^2 + 3ab\cos\theta}\, d\theta$.

The result follows from observing that $a > 2b$, $A = \pi(a^2 + \frac{1}{2}b^2)$, and

$$\int_0^\pi \frac{\cos^2\theta\, d\theta}{A + B\cos\theta} = A\int_0^\pi \frac{\cos^2\theta\, d\theta}{A^2 - B^2\cos^2\theta}.$$

5330. (By the late Professor CLIFFORD, F.R.S.)—Show that

$$\int_0^{\frac{1}{2}\pi} \cos(\alpha \tan x)\, e^{\beta \tan x}\, dx = \tfrac{1}{2}\pi e^{-\alpha} (\cos\beta + \sin\beta).$$

Note by J. J. WALKER, M.A., F.R.S.

It may be of interest to show that this Question, proposed as far back as 1863, is erroneous; nor is it likely that such an integral could be evaluated in a finite form. Assume $\tan x = y$, so that the given integral becomes

$$u = \int_0^\infty \cos(\alpha y)\, e^{\beta y} \div (1+y^2)\,.\,dy. \quad \text{Then } u - \frac{d^2u}{d\alpha^2} = \int_0^\infty \cos\alpha y\,.\,e^{\beta y}\, dy.$$

For the value of u as given in the Question, the sinister of this equality is zero; but the dexter, a well-known integral, is infinite for β positive, and equal to $\beta \div (\alpha^2+\beta^2)$ for β negative.

7076. (By Professor TOWNSEND, F.R.S.)—Two circular cylinders round axes passing through the point of no linear acceleration O of a rigid body in motion, in directions parallel to those of the angular velocity and of the angular acceleration, at any instant of the motion, being supposed described through any arbitrary point P of the body; show that the entire linear acceleration of P, at the instant, consists of two distinct components, due respectively to angular velocity and to angular acceleration, the former normal to the first and the latter tangential to the second of the two aforesaid cylinders, and each directly proportional to the radius of its cylinder.

Solution by the PROPOSER.

Denoting by u, v, w the components of the linear velocity of the centre of inertia (or of any other definite point) $\bar{x}\,\bar{y}\,\bar{z}$ of the body, and by p, q, r those of the angular velocity of its entire mass, at any instant of the motion; then, since for the components of the linear velocity of any other point xyz of the mass at the instant,

$$\frac{dx}{dt} = u + q(z-\bar{z}) - r(y-\bar{y}), \quad \frac{dy}{dt} = v + \&c., \quad \frac{dz}{dt} = w + \&c.;$$

therefore, for those of the linear acceleration of xyz at the instant,

$$\frac{d^2x}{dt^2} = \frac{du}{dt} - (q^2+r^2)(x-\bar{x}) + pq(y-\bar{y}) + pr(z-\bar{z})$$
$$+ \frac{dq}{dt}(z-\bar{z}) - \frac{dr}{dt}(y-\bar{y}),$$

with similar values for $\dfrac{d^2y}{dt^2}$ and for $\dfrac{d^2z}{dt^2}$, the dexters of which equated to 0 give at once three independent linear equations for the determination of $(x_0-\bar{x})$, $(y_0-\bar{y})$, $(z_0-\bar{z})$, and therefore of x_0, y_0, z_0, for the point O of no linear acceleration at the instant; and show that for every instant there

is always one, and generally but one, such point in the space of the motion, not necessarily included in, but definitely connected with, the body's mass.

Denoting now by ξ, η, ζ the relative coordinates $(x-x_0)$, $(y-y_0)$, $(z-z_0)$ with respect to O of any other point P of the mass at the instant, and substituting in the dexters of the preceding equations for x, y, z their equivalents $(x_0+\xi)$, $(y_0+\eta)$, $(z_0+\zeta)$, we get from them immediately, by virtue of the three aforesaid evanescences characteristic of the position of O, for the components of the linear acceleration of P at the instant,

$$\frac{d^2x}{dt^2} = -(q^2+r^2)\,\xi + pq\eta + rp\zeta + \frac{dq}{dt}\zeta - \frac{dr}{dt}\eta,$$

$$\frac{d^2y}{dt^2} = -(r^2+p^2)\,\eta + qr\zeta + pq\xi + \frac{dr}{dt}\xi - \frac{dp}{dt}\zeta,$$

$$\frac{d^2z}{dt^2} = -(p^2+q^2)\,\zeta + rp\xi + qr\eta + \frac{dp}{dt}\eta - \frac{dq}{dt}\xi,$$

which show that, as stated in the question, the entire linear acceleration of P at the instant consists of two distinct parts, one depending on p, q, r, and directed normally to the cylinder of radius ρ through x, y, z, whose equation to origin O is

$$(q\zeta-r\eta)^2+(r\xi-p\zeta)^2+(p\eta-q\xi)^2 = (p^2+q^2+r^2)\,\rho^2,$$

and the other depending on $\frac{dp}{dt}$, $\frac{dq}{dt}$, $\frac{dr}{dt}$, or p', q', r', and directed tangentially to the cylinder of radius ρ' whose equation to same origin is

$$(q'\zeta-r'\eta)^2+(r'\xi-p'\zeta)^2+(p'\eta-q'\xi)^2 = (p'^2+q'^2+r'^2)\,\rho'^2,$$

and each component acting perpendicularly to the axis, and varying directly as the radius of its cylinder.

Expressed in terms of the radii ρ and ρ' of the two cylinders, and of the components pqr and $p'q'r'$ of the angular velocity and angular acceleration of the mass at the instant, the values of the two parts of the linear acceleration at xyz are respectively

$$(p^2+q^2+r^2)\,\rho \text{ and } \pm\,(p'^2+q'^2+r'^2)^{\frac{1}{2}}\,\rho\,;$$

as is evident from the above equations.

If at any instant of the motion the axes of angular velocity and of angular acceleration of the mass happen to coincide in the space of the motion, that is, if at any instant $p' : q' : r' = p : q : r$, the two aforesaid cylinders corresponding to them coincide also completely at the instant, for every point of the body; and the linear acceleration from point to point of the mass follows consequently the same simple law as for the motion of a lamina in its plane,—there being then, in fact, an *axis*, in place of, as usual, only a *centre*, of no linear acceleration of the motion.

[Professor Minchin remarks that the theorem here enunciated was published by him in *Nature* (for November 18, 1880), and that he had arrived at the result for uniplanar motion, and mentioned it to Professor Wolstenholme, who communicated to him the extension to three-dimensional motion. The general result was known previously, and is to be found in the works of some foreign authors. In Minchin's *Uniplanar Kinematics*, the theorem has been made use of for the solution of several problems.]

7332. (By the EDITOR.)—If p_1, p_2, p_3 be the perpendiculars from the vertices of a triangle on the opposite sides; d_1, d_2, d_3 the distances from the vertices to the points of contact of the escribed circles with the opposite sides; and $l_1^2 = d_1 + r_1^2$, $l_2^2 =$ &c., $l_3^2 =$ &c.; prove that

$$\frac{l_1^2}{p_1r_1} = \frac{l_2^2}{p_2r_2} = \frac{l_3^2}{p_3r_3} = \frac{2}{r}(\mathrm{R}-r), \quad \frac{l_1^2}{bcr_1} = \frac{l_2^2}{car_2} = \frac{l_3^2}{abr_3} = \frac{1}{r} - \frac{1}{\mathrm{R}}.$$

Solution by G. HEPPEL, M.A.; Prof. MATZ, M.A.; *and others.*

Let D be the foot of the perpendicular from A (the figure may be easily imagined); E the point of contact with BC. Then $\mathrm{CE} = s-b$, $\mathrm{CD} = b\cos\mathrm{C}$;

therefore $$\mathrm{DE} = \frac{a^2+b^2-c^2}{2a} - \frac{a+c-b}{2} = \frac{(b-c)\,s}{2},$$

$$\mathrm{DE}^2 = \frac{(b-c)^2 s^2}{a^2} = \frac{[a^2-4\,(s-b)(s-c)]\,s^2}{a^2} = s^2 - \frac{4\Delta^2 . s}{a^2\,(s-a)} = s^2 - p_1^2 - \frac{4\Delta^2}{a\,(s-a)}$$

hence we have $l_1^2 = \mathrm{DE}^2 + p_1^2 + r_1^2 = s^2 - \dfrac{4\Delta}{a\,(s-a)} + \dfrac{\Delta^2}{(s-a)^2}$,

therefore $$\frac{l_1^2}{p_1r_1} = \frac{s^2\,a\,(s-a)}{2\Delta^2} - 2 + \frac{a}{2\,(s-a)} = \frac{as\,[s\,(s-a)+(s-b)(s-c)]}{2\Delta^2} - 2$$

$$= \frac{abcs}{2\Delta^2} - 2 = \frac{2\mathrm{R}}{r} - 2 = \frac{2}{r}(\mathrm{R}-r),$$

therefore $$\frac{l_1^2}{bcr_1} = \frac{p_1}{bc} \cdot \frac{2}{r} \cdot (\mathrm{R}-r) = \frac{2\sin\mathrm{C}}{rc} \cdot (\mathrm{R}-r) = \frac{1}{r} - \frac{1}{\mathrm{R}}.$$

[If O_1 be the centre of the circle escribed to the side a, we have

$$\mathrm{AO}_1 = s\sec\tfrac{1}{2}\mathrm{A}, \quad r_1 = s\tan\tfrac{1}{2}\mathrm{A}, \quad p_1 = c\sin\mathrm{B} = \frac{bc}{2\mathrm{R}};$$

$$\frac{\mathrm{AO}_1^2}{p_1r_1} = \frac{2s}{p_1\sin\mathrm{A}} = \frac{2\Delta}{p_1r\sin\mathrm{A}} = \frac{4\mathrm{R}\Delta}{p_1ar} = \frac{2\mathrm{R}}{r}; \quad l_1^2 = \mathrm{AO}_1^2 - 2p_1r_1;$$

$$\frac{l_1^2}{p_1r_1} = \frac{2\mathrm{R}}{r} - 2 = \frac{2}{r}(\mathrm{R}-r); \quad \frac{l_1^2}{bcr_1} = \frac{l_1^2}{2\mathrm{R}p_1r_1} = \frac{1}{r} - \frac{1}{\mathrm{R}}.]$$

7309 & 7362. (By W. J. C. SHARP, M.A., and G. HEPPEL, M.A.)—If S_r denote the sum of the r^{th} homogeneous products of any quantities; s_r the sum of the r^{th} powers of the same quantities; and p_r the sum of the combinations taken r together; prove that

$$\left.\begin{array}{c} \mathrm{S}_1 = s_1, \; 2\mathrm{S}_2 = \mathrm{S}_1 s_1 + s_2, \; 3\mathrm{S}_3 = \mathrm{S}_2 s_1 + \mathrm{S}_1 s_2 + s_3 \\ r\mathrm{S}_r = \mathrm{S}_{r-1} s_1 + \mathrm{S}_{r-2} s_2 + \mathrm{S}_{r-3} s_3 + \ldots + s_r \end{array}\right\} \ldots\ldots(7309),$$

$$\mathrm{S}_r = \mathrm{S}_{r-1} . p_1 - \mathrm{S}_{r-2}\, p_2 + \mathrm{S}_{r-3} . p_3 \ldots\ldots \pm p_r \ldots\ldots\ldots\ldots(7362).$$

Solution by G. HEPPEL, M.A.

Let $a, b, c, d \dots k, l$ be the quantities; S_r the sum of r^{th} homogeneous products; s_r the sum of r^{th} powers; p_r the sum of combinations taken r together. Let $u = (1+ax+a^2x^2+\dots)(1+bx+b^2x^2+\dots)\dots$ or its equivalent $\dfrac{1}{1-p_1x+p_2x^2-\dots}$; $v = p_1x-p_2x^2+\dots$; so that $u = \dfrac{1}{1-v}$ and $u = uv+1$; u_1, v_1, u_2, v_2, &c., be the successive differential coefficients of u and v with respect to x; and U_1V_1, U_2V_2, &c. be the values of these when $x=0$. Then S_r is the coefficient of x^r in u; therefore $S_r = U_r \div r!$

Also $\quad u_r = u_r v + r u_{r-1} v_1 + \frac{1}{2} r(r-1) u_{r-2} \cdot v_2 + \dots\dots + u v_r.$

Put $x = 0$, and remember that $V_r = \pm r! \cdot p_r$,

then $\quad U_r = rU_{r-1} \cdot p_1 - r(r-1) U_{r-2} \cdot p_2 + \dots \pm r!\, p_r,$

therefore $\quad S_r = S_{r-1} p_1 - S_{r-2} \cdot p_2 + S_{r-3} \cdot p_3 - \dots \pm p_r.$

In TODHUNTER'S *Theory of Equations* and elsewhere, it is proved that, $s_r = s_{r-1}p_1 - s_{r-2}p_2 + s_{r-3}p_3 - \dots \pm rp_r$; but the following new and shorter proof depends only on first principles:—

$$s_r = p_1 s_{r-1} - \Sigma(a^{r-1} \cdot b), \quad -\Sigma(a^{r-1} \cdot b) = -p_2 s_{r-2} + \Sigma(a^{r-2} \cdot bc),$$
$$\Sigma(a^{r-2} \cdot bc) = p_3 s_{r-3} - \Sigma(a^{r-3} \cdot bcd), \text{ \&c., \&c.,}$$
$$\mp \Sigma(a^2 bcd \dots k) = \mp p_{r-1} s_1 \pm rp_r;$$

therefore, adding the equations, $s_r = s_{r-1}p_1 - s_{r-2} \cdot p_2 + s_{r-3}p_3 - \dots \pm rp_r$.
The series in 7309 is derived from these, as follows:—

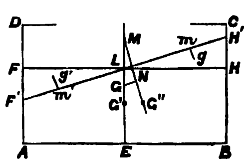
It is evidently true for the first few terms. Assume it to be true as far as $(r-1)$ terms; so that $(r-1)S_{r-1} - s_{r-1} = S_{r-2}s_1 + S_{r-3}s_2 + \dots + S_1 s_{r-2}$. Then, from the series found above, we obtain

$$\begin{aligned} rS_r - s_r &= (rS_{r-1} - s_{r-1})p_1 - (rS_{r-2} - s_{r-2})p_2 + \dots \pm (rS_1 - s_1)p_{r-1} \\ &= S_{r-1} \cdot p_1 - 2S_{r-2} \cdot p_2 + 3S_{r-3} \cdot p_3 - 4S_{r-4} \cdot p_4 + \dots \pm (r-1)S_1 p_{r-1} \\ &\quad + (S_{r-2} \cdot s_1 + S_{r-3} \cdot s_2 + S_{r-4} s_3 + \dots + S_1 \cdot s_{r-2})p_1 \\ &\quad - (S_{r-3} \cdot s_1 + S_{r-4} \cdot s_2 + S_{r-5} \cdot s_3 + \dots + S_1 \cdot s_{r-3})p_2 + \dots \pm S_1 p_{r-1}. \end{aligned}$$

Hence, collecting coefficients of S_{r-1}, S_{r-2}, &c., and using the second series, $\quad rS_r - s_r = S_{r-1} \cdot s_1 + S_{r-2} \cdot s_2 + S_{r-3} \cdot s_3 + \dots + S_1 s_{r-1}.$

7378. (By Professor HAUGHTON, F.R.S.)—A homogeneous rectangular parallelepiped, the edges of which are a, b, c, floats in a liquid whose density is ρ; and is turned through an angle θ (the top remaining above the surface of the liquid), so that the plane of a, b remains parallel to itself; find the limiting value of θ, when the solid will cease to right itself.

Solution by R. RAWSON; SARAH MARKS; *and others.*



Let ABCD be a vertical section through the centre of gravity (G) of the parallelepiped. The liquid lines FH, F'H' before and after the solid is turned through an angle θ, are determined—(1) by the equality of the weights of the displaced liquid and parallelepiped; (2) by the equality of the weights of the *in* and *out*. Hence $EL = aw : w'$,

where w, w' are the weights of the cubic units of the solid and liquid respectively. Bisect EL in G′, which is the centre of gravity of the displaced liquid. Let G″ be the centre of gravity of the displaced liquid, after the solid has been turned through the angle θ.

Draw G″M perpendicular to F′H′, meeting EL produced in M, which is called by Atwood and others the metacentre corresponding to the definite angle θ. When, however, the angle θ is indefinitely diminished, the point M becomes the metacentre first assigned by Bouguer. Let g, g' be the centre of gravity of the *in* and *out*, respectively; draw gm, $g'm'$ perpendicular to the liquid line F′H′ and GN perpendicular to G″M.

Now, $abcw$. GN = the righting moment, or the force by which the solid endeavours to gain its upright position.

Atwood has shown that $\mathrm{GN} = \dfrac{mm'.\,v}{\mathrm{V}} - \mathrm{GG'}\sin\theta$ ....(1),

where V, v are the volumes of immersion and *in* and *out* respectively. (See Fincham's *Outlines of Shipbuilding*, p. 152); and it is readily seen that

$$\mathrm{V} = \frac{abcw}{w'}, \quad \text{and} \quad v = \frac{b^2c\tan\theta}{8}, \quad \mathrm{GG'} = \frac{a}{2} - \frac{aw}{2w'} = \frac{a(w'-w)}{2w'},$$

therefore $$\frac{v}{\mathrm{V}} = \frac{b^2c\tan\theta \times w'}{8\,.\,abcw} = \frac{bw'\tan\theta}{8aw};$$

$$mm' = 2\mathrm{L}m = 2\left[\tfrac{1}{3}\mathrm{LH'} + \tfrac{1}{3}(\mathrm{LH}\cos\theta - \tfrac{1}{3}\mathrm{LH'})\right] = \tfrac{4}{9}\mathrm{LH'} + \mathrm{LH}\cos\theta);$$

but $2\mathrm{LH'} = b\sec\theta$, and $2\mathrm{LH} = b$, $\therefore$ $mm' = \frac{1}{3}b(\sec\theta + \cos\theta)$.

Substituting in (1),

$$\mathrm{GN} = \frac{\sin\theta}{2}\left\{\frac{b^2w'}{12aw}\left(1 + \frac{1}{\cos^2\theta}\right) - \frac{a(w'-w)}{w'}\right\} \text{............(2)}$$

and $$\mathrm{GM} = \tfrac{1}{2}\left\{\frac{b^2w'}{12aw}\left(1 + \frac{1}{\cos^2\theta}\right) - \frac{a(w'-w)}{w'}\right\} \text{..................(3).}$$

The solid will cease to right itself when GN equals zero,

or $$\cos\theta = \frac{bw'}{(12a^2ww' - 12a^2w^2 - b^2w'^2)^{\frac{1}{2}}},$$

a value which is possible when $\left(\dfrac{w}{w'}\right)^2 - \dfrac{w}{w'} + \dfrac{b^2}{12a^2} < 0$.

7397. (By C. Bickerdike.)—A point moves with constant velocity in a straight line, prove that its angular velocity about any fixed point is inversely as the square of the distance from this point.

Solution by G. S. Carr, B.A.

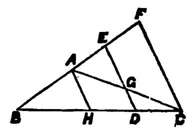
Let p = perpendicular on the line; then, geometrically, we have

$$\frac{r\,d\theta}{dx} = \frac{p}{r}; \text{ therefore } \frac{d\theta}{dt} = \frac{p}{r^2}\frac{dx}{dt} = \frac{pv}{r^2}.$$

[For another proof, see also Thomson and Tait's *Nat. Phil.*, p. 30.]

7398. (By R. Knowles, B.A.)—In the side BC (> AB) of a triangle ABC, BD is taken equal to one-half of AB + BC; and in BA produced, BE is taken equal to BD; prove that DE bisects AC at G.

Solution by Kate Gale; S. Greenidge; *and others.*



Make EF = AE and join FC; then EF = AE = BD − AB = ½ (AB + BC) − AB = ½ (BC − AB); DC = BC − ½ (AB + BC) = ½ (BC − AB); therefore AE = EF = DC; hence CF is parallel to GE. and as AE = EF, therefore AG = GC.

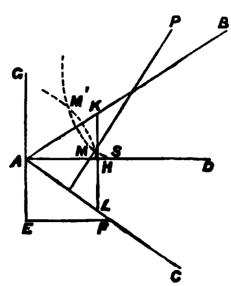
[If AH be drawn parallel to DGE, we have

$$BD = \tfrac{1}{2}(BH + BC) = BH + \tfrac{1}{2}HC;$$

therefore HD = DC, and consequently AG = GC.]

7336 & 7369. (By W. H. Blythe, M.A., and A. H. Curtis, LL.D.)—Through a given point to draw a straight line which shall (7336) bisect a given triangle, (7369) form with two given straight lines a given area.

Solution by G. Heppel, M.A.; Professor Matz, M.A.; *and others.*



(7369).—Let AB, AC be the given lines, and first let the given point P be outside BAC. Let AD be the bisector; AE, perpendicular to AD, a side of the square equal to the given area. Draw EF parallel to AD. Make AG = EF. Let the circle with diameter EG cut AD in H. Draw KHL perpendicular to AD, and in AD make AS = AK. Let the circle whose radius is AH cut the circle whose diameter is SP in M and M′. Then one of the lines PM, PM′ will be the line required.

For $\quad AH^2 = AE \,.\, AG = AE \,.\, EF,$

and $\quad HK : AH = AE : EF;$

$\therefore\;$ AH . HK or Δ KAL = AE^2 = given area.

Now S is the focus of a hyperbola, whose major axis is twice AH, and whose asymptotes are AB and AC; and the latter part of the construction is merely the ordinary method of drawing a tangent from P to such a hyperbola. This must cut off from the angle BAC the area required. If P is inside the angle, the construction is the same, but both lines PM, PM′, will give a solution.

(7336).—If A be one angle of the triangle ABC, we have merely to take AK a mean proportional to AB and ½AC; and proceed as before.

7304. (By Professor WOLSTENHOLME, M.A.)—O is the centre of the circle ABC, O′ the point of concourse of the three straight lines each joining an angular point of the triangle ABC to the common point of the tangents, at the ends of the opposite side, to the circle ABC [or the second focus of the ellipse inscribed in the triangle ABC and having one focus at the centroid]; prove that (1) the two points P, defined by the equations AP . BC = BP . CA = CP . AB, lie on the straight line OO′, and divide it in the ratios $1+\cos A\cos B\cos C : \pm\sqrt{3}\sin A\sin B\sin C$; also (2) if from either point P perpendiculars be drawn on the sides of the triangle ABC, the triangle formed by joining the feet of these perpendiculars will be equilateral. [If each angle A, B, C be $< 120°$, the angles subtended will be $A+60°$, the triangle ABC at one point P (between O and O′) by the sides of $B+60°$, $C+60°$; at the other point they will be

either $\quad A-60°,\ B-60°,\ 60°-C,$ if $A>60°$ and $B>60°$,

or $\quad 60°-A,\ 60°-B,\ C-60°,$ if $A<60°$ and $B<60°$.]

Solution by W. S. M'CAY, M.A.; D. EDWARDES; *and others.*

If r_1, r_2, r_3 be the distances of a point P from the vertices of a triangle ABC, the feet of perpendiculars on the sides from P are vertices of a triangle A′B′C′ whose sides are $r_1\sin A$, $r_2\sin B$, $r_3\sin C$.

This triangle is well known to be a minimum of its species inscribed in ABC; and in Question 7067 I have shown that there are two such triangles, the corresponding positions of P being inverses to the circumscribed circle (S) of ABC. In the present case, where $ar_1 = br_2 = cr_3$, the triangle A′B′C′ is equilateral (2), and P is determined by the intersections of three coaxal circles. Take one of these circles, $ar_1 = br_2$. This circle cuts S orthogonally at C and has its centre on AB, therefore the pole of O to this circle is the line joining C to the intersection of tangents to S at AB. And, O being on the radical axis of the three circles, its three polars concur at another point on the radical axis, and I have just shown that this is the point O′ in the question.

To find the ratio in which OO′ is cut by any of the circles, suppose $ar_1 = br_2$, or in trilinear coordinates $x^2-y^2+2z(x\cos B-y\cos A)=0$; the coordinates of OO′ are $\cos A : \cos B : \cos C$, $\sin A : \sin B : \sin C$. These have to be multiplied by θ and ϕ to represent the actual perpendiculars on the sides, where $\dfrac{\theta}{\phi} = \dfrac{1+\cos A\cos B\cos C}{\sin A\sin B\sin C}$, and substituting $\lambda\theta\cos A+\mu\phi\sin A$, &c., for x, y, z, we get $\lambda^2\theta^2 = 3\mu^2\phi^2$, the result.

It is not hard to see that the line joining C to the pole of AB to S makes the same angles with the adjacent sides as the bisector of the base (draw a parallel through C to AB to meet S in C′, and join C′ to polar of AB), and so the centroid and O′ are foci of an ellipse inscribed in ABC.

As to the angles subtended at P by the sides of ABC, it is shown in TOWNSEND'S *Modern Geometry*, Vol. i., p. 56, that these angles are sums or differences of corresponding angles of ABC and A′B′C′. It will be found that they are sums if P be *inside* the circumscribed circle of ABC, and differences if P be *outside*; and the two positions of P, being inverses to S, the angles will be sums for one of them and differences for the other. If the feet of perpendiculars from P be vertices of a triangle whose angles are A′, B′, C′, the coordinates of the common conjugate of O to the three circles that determine P may similarly be shown to be

$$\sin A . \cot A' : \sin B . \cot B' : \sin C . \cot C'.$$

6670. (By Belle Easton.)—Through a given point P, between two given lines AB, AC, draw a straight line BPC meeting the given lines in B and C, so that BPC may be a minimum.

Solution by G. Heppel, M.A.; J. O'Regan; *and others.*

Take AB and AC as axes. Let $AB = h$, $AC = k$, and let the point P be (a, b); then, since P is on BC, $\dfrac{a}{h} + \dfrac{b}{k} = 1$, therefore $k = \dfrac{bh}{h-a}$, and if l be the length of BC, $l^2 = h^2 + \dfrac{b^2h^2}{(h-a)^2} - \dfrac{2bh^2}{h-a}.\cos A$. Put $x = h-a$

$$\text{and } c = b\cos A, \text{ then } l^2 = (x+a)^2 + \frac{b^2(x+a)^2}{x^2} - \frac{2c(x+a)^2}{x};$$

therefore
$$l^2 = (x+a)^2\left(1 + \frac{b^2}{x^2} - \frac{2c}{x}\right);$$

therefore, if l is a minimum, we have

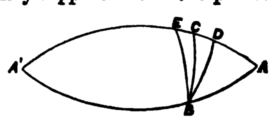
$$(x+a)\left(\frac{c}{x^2} - \frac{b^2}{x^3}\right) + 1 + \frac{b^2}{x^2} - \frac{2c}{x} = 0, \text{ or } x^3 - cx^2 + acx - ab^2 = 0,$$

a cubic equation which determines x. If the lines are at right angles, $c = 0$ and $x = \sqrt[3]{ab^2}$, whence by substitution $l^2 = (a^{\frac{2}{3}} + b^{\frac{2}{3}})^3$.

5980. (By Professor Seitz, M.A.)—Three points, taken at random in the surface of a sphere, are joined by arcs of great circles; show that the chance (1) that the triangle formed has all its angles acute, is $\dfrac{1}{2\pi} - \dfrac{1}{8}$; (2) that it has one obtuse angle, is $\dfrac{9}{8} - \dfrac{3}{2\pi}$; (3) that it has two obtuse angles, is $\dfrac{3}{2\pi} - \dfrac{3}{8}$; and (4) that it has all its angles obtuse, is $\dfrac{3}{8} - \dfrac{1}{2\pi}$.

Solution by the Proposer.

Let A, B, C be the random points. We may suppose one of the points, as A, fixed. Produce the arcs AB and AC till they meet at A′; draw the arcs BD and BE perpendicular, respectively, to ACA′ and ABA′.



Let $\angle BAC = \theta$, arc $AB = \mu$, $AC = \phi$, $AD = \phi_1$, $AE = \phi_2$, $BD = \psi$, $BE = \omega$, and take unity for the radius of the sphere. Then it is easily shown that $\cos\phi_1 \sin\mu\, d\mu = \operatorname{cosec}^2\theta \sin\psi\, d\psi$, and $\cos\phi_2 \sin\mu\, d\mu = \cot^2\theta \sin\omega \sec^2\omega\, d\omega$.

An element of surface at B is $2\pi \sin\mu\, d\mu$, and at C it is $d\theta \sin\phi\, d\phi$.

If $\theta < \frac{1}{2}\pi$ and $\phi < \frac{1}{2}\pi$, or if $\theta > \frac{1}{2}\pi$ and $\phi > \frac{1}{2}\pi$, E will lie in DA′; but if $\theta < \frac{1}{2}\pi$ and $\phi > \frac{1}{2}\pi$, or if $\theta > \frac{1}{2}\pi$ and $\phi < \frac{1}{2}\pi$, E will lie in AD.

1. If $\theta < \frac{1}{2}\pi$, $\phi < \frac{1}{2}\pi$, and C lies in DE, the triangle will have all its

angles acute. Hence we have

$$p = \frac{1}{8\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{\phi_1}^{\phi_2} d\theta\,.\,2\pi\sin\mu\,d\mu\sin\phi\,d\phi$$

$$= \frac{1}{4\pi}\int_0^{\frac{1}{2}\pi}\left\{\int_0^{\theta}\operatorname{cosec}^2\theta\sin\psi\,d\psi - \int_0^{\theta}\cot^2\theta\sin\omega\sec^2\omega\,d\omega\right\}d\theta$$

$$= \frac{1}{4\pi}\int_0^{\frac{1}{2}\pi}\tan^2\tfrac{1}{2}\theta\,d\theta = \frac{1}{2\pi} - \frac{1}{8}.$$

2. If $\theta < \frac{1}{2}\pi$, $\phi < \frac{1}{2}\pi$, and C lies in AD or EA′; or if $\theta < \frac{1}{2}\pi$, $\phi > \frac{1}{2}\pi$, and C lies in AE or DA′; or if $\theta > \frac{1}{2}\pi$, $\phi < \frac{1}{2}\pi$, and C lies in AE; or if $\theta > \frac{1}{2}\pi$, $\phi > \frac{1}{2}\pi$, and C lies in AD; the triangle will have one obtuse angle. Hence we have

$$p_1 = \frac{1}{8\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\left\{\int_0^{\phi_1}\sin\phi\,d\phi + \int_{\phi_2}^{\pi}\sin\phi\,d\phi\right\}d\theta\,.\,2\pi\sin\mu\,d\mu$$

$$+ \frac{1}{8\pi^2}\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi}^{\pi}\left\{\int_0^{\phi_1}\sin\phi\,d\phi + \int_{\phi_1}^{\pi}\sin\phi\,d\phi\right\}d\theta\,.\,2\pi\sin\mu\,d\mu$$

$$+ \frac{1}{8\pi^2}\int_{\frac{1}{2}\pi}^{\pi}\left\{\int_0^{\frac{1}{2}\pi}\int_0^{\phi_2}2\pi\sin\mu\,d\mu\sin\phi\,d\phi + \int_{\frac{1}{2}\pi}^{\pi}\int_0^{\phi_1}2\pi\sin\mu\,d\mu\sin\phi\,d\phi\right\}d\theta$$

$$= \frac{1}{2\pi}\int_0^{\frac{1}{2}\pi}\left\{\int_0^{\pi}\sin\mu\,d\mu - \int_0^{\theta}\operatorname{cosec}^2\theta\sin\psi\,d\psi + \int_0^{\theta}\cot^2\theta\sin\omega\sec^2\omega\,d\omega\right\}d\theta$$

$$+ \frac{1}{4\pi}\int_{\frac{1}{2}\pi}^{\pi}\left\{\int_0^{\pi}\sin\mu\,d\mu - \int_{\theta}^{\pi}\operatorname{cosec}^2\theta\sin\psi\,d\psi + \int_{\theta}^{\pi}\cot^2\theta\sin\omega\sec^2\omega\,d\omega\right\}d\theta$$

$$= \frac{1}{2\pi}\int_0^{\frac{1}{2}\pi}(3-\sec^2\tfrac{1}{2}\theta)\,d\theta + \frac{1}{4\pi}\int_{\frac{1}{2}\pi}^{\pi}(3-\operatorname{cosec}^2\tfrac{1}{2}\theta)\,d\theta = \frac{9}{8} - \frac{3}{2\pi}.$$

3. If $\theta < \frac{1}{2}\pi$, $\phi > \frac{1}{2}\pi$, and C lies in ED; or if $\theta > \frac{1}{2}\pi$, $\phi < \frac{1}{2}\pi$, and C lies in ED; or if $\theta > \frac{1}{2}\pi$, $\phi > \frac{1}{2}\pi$, and C lies in DE; the triangle will have two obtuse angles. Hence we have

$$p_2 = \frac{1}{8\pi^2}\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi}^{\pi}\int_{\phi_2}^{\phi_1} d\theta\,.\,2\pi\sin\mu\,d\mu\sin\phi\,d\phi$$

$$+ \frac{1}{8\pi^2}\int_{\frac{1}{2}\pi}^{\pi}\left\{\int_0^{\frac{1}{2}\pi}\int_{\phi_2}^{\phi_1}2\pi\sin\mu\,d\mu\sin\phi\,d\phi + \int_{\frac{1}{2}\pi}^{\pi}\int_{\phi_1}^{\phi_2}2\pi\sin\mu\,d\mu\sin\phi\,d\phi\right\}d\theta$$

$$= \frac{1}{4\pi}\int_0^{\frac{1}{2}\pi}\left\{\int_0^{\theta}\operatorname{cosec}^2\theta\sin\psi\,d\psi - \int_0^{\theta}\cot^2\theta\sin\omega\sec^2\omega\,d\omega\right\}d\theta$$

$$+ \frac{1}{2\pi}\int_{\frac{1}{2}\pi}^{\pi}\left\{\int_{\theta}^{\pi}\operatorname{cosec}^2\theta\sin\psi\,d\psi - \int_{\theta}^{\pi}\cot^2\theta\sin\omega\sec^2\omega\,d\omega\right\}d\theta$$

$$= \frac{1}{4\pi}\int_0^{\frac{1}{2}\pi}\tan^2\tfrac{1}{2}\theta\,d\theta + \frac{1}{2\pi}\int_{\frac{1}{2}\pi}^{\pi}\cot^2\tfrac{1}{2}\theta\,d\theta = \frac{3}{2\pi} - \frac{3}{8}.$$

4. If $\theta > \frac{1}{2}\pi$, $\phi < \frac{1}{2}\pi$, and C lies in DA′; or if $\theta > \frac{1}{2}\pi$, $\phi > \frac{1}{2}\pi$, and C lies in EA′; the triangle will have all its angles obtuse. Hence we have

$$p_3 = \frac{1}{8\pi^2}\int_{\frac{1}{2}\pi}^{\pi}\left\{\int_0^{\frac{1}{2}\pi}\int_{\phi_1}^{\pi} 2\pi \sin\mu\, d\mu \sin\phi\, d\phi + \int_{\frac{1}{2}\pi}^{\pi}\int_{\phi_2}^{\pi} 2\pi \sin\mu\, d\mu \sin\phi\, d\phi\right\} d\theta$$

$$= \frac{1}{4\pi}\int_{\frac{1}{2}\pi}^{\pi}\left\{\int_0^{\pi}\sin\mu\, d\mu - \int_{\theta}^{\pi}\operatorname{cosec}^2\theta \sin\psi\, d\psi + \int_{\theta}^{\pi}\cot^2\theta \sin\omega \sec^2\omega\, d\omega\right\} d\theta$$

$$= \frac{1}{4\pi}\int_{\frac{1}{2}\pi}^{\pi}\left(3 - \operatorname{cosec}^2 \tfrac{1}{2}\theta\right) d\theta = \frac{3}{8} - \frac{1}{2\pi}.$$

6833. (By the Editor.) — Show that the volume between $x = 0$ and $x = 2l$ of the solid bounded by the surface whose equation is

$$a\,(y^4 - z^4) - z^2\,(x^3 - 2ay^2 + 2c^3) - y^2\,(bx^2 + c^2x + e^3) = 0,$$

is

$$\frac{\pi}{3a}\,(6l^4 + 4bl^3 + 3c^2l^2 + 9e^3l).$$

Solution by D. Edwardes; G. Eastwood, M.A.; *and others.*

While x is constant, the equation of the section is

$$r^2 = \mathrm{A}\cos^2\theta + \mathrm{B}\sin^2\theta \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where $a\mathrm{A} = bx^2 + c^2x + e^3$, and $a\mathrm{B} = x^3 + 2e^3$. The area of (1) is $\frac{1}{2}\pi\,(\mathrm{A}+\mathrm{B})$; therefore the required volume is

$$\frac{\pi}{2a}\int_0^{2l}(x^3 + bx^2 + c^2x + 3e^3)\,dx = \frac{\pi}{6a}\,(12l^4 + 8bl^3 + 6c^2l^2 + 18e^3l) = \&c.$$

6739. (By Professor Wolstenholme, M.A.)—If $u^2 = 0$ be the rational equation of the second degree of a conic referred to Cartesian coordinates inclined at an angle ω, prove that the equations giving (1) the foci, (2) the director circle, (3) all four directrices, are

$$\frac{d^2u}{dx^2} = \frac{d^2u}{dy^2} = \frac{d^2u}{dx\,dy}\sec\omega,\quad \frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 2\,\frac{d^2u}{dx\,dy}\cos\omega \;\ldots\ldots\; (1,\,2);$$

$$\left\{\frac{du}{dx}\frac{du}{dy}\cos 2\omega\left[\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2\right]\cos\omega - u\,\frac{d^2u}{dx\,dy}\right\}^2$$

$$= \left\{2\frac{du}{dx}\frac{du}{dy}\cos\omega + \left(\frac{du}{dx}\right)^2 - u\,\frac{d^2u}{dx^2}\right\}\left\{2\frac{du}{dx}\frac{du}{dy}\cos\omega + \left(\frac{du}{dx}\right)^2 - u\,\frac{d^2u}{dy^2}\right\} \ldots (3).$$

Solution by the Proposer.

If $u^2 = 0$ be the rational equation of a conic referred to Cartesian coordinates inclined at an angle ω, and we move the origin to (x, y) a focus,

using (X, Y) for current coordinates, the equation will become of the form $\quad k(X^2+Y^2+2XY\cos\omega) = (pX+qY+r)^2$;
but u^2 will become

$$u^2 + X\,.\,2u\frac{du}{dx} + Y\,.\,2u\frac{du}{dy} + X^2\left\{\left(\frac{du}{dx}\right)^2 + u\frac{d^2u}{dx^2}\right\} + Y^2\left\{\left(\frac{du}{dy}\right)^2 + u\frac{d^2u}{dy^2}\right\}$$
$$+ 2XY\left(\frac{du}{dx}\frac{du}{dy} + u\frac{d^2u}{dx\,dy}\right) = 0\,;$$

and, making this coincide with the former,

$$u^2 = \lambda r^2,\quad u\frac{du}{dx} = \lambda pr,\quad u\frac{du}{dy} = \lambda qr,$$
$$\left(\frac{du}{dx}\right)^2 + u\frac{d^2u}{dx^2} = \lambda(p^2-k),\quad \left(\frac{du}{dy}\right)^2 + u\frac{d^2u}{dy^2} = \lambda(q^2-k),$$
$$\frac{du}{dx}\frac{du}{dy} + u\frac{d^2u}{dx\,dy} = \lambda(pq - k\cos\omega)\,;$$

whence $\quad \lambda p^2 = \left(\frac{du}{dx}\right)^2,\quad u\frac{d^2u}{dx^2} = -\lambda k\,;\quad \lambda q^2 = \left(\frac{du}{dy}\right)^2,\quad u\frac{d^2u}{dy^2} = -\lambda k\,;$

$$\lambda pq = \frac{du}{dx}\frac{du}{dy},\quad u\frac{d^2u}{dx\,dy} = -\lambda k\cos\omega\,;$$

or the equations for the foci will be

$$\frac{d^2u}{dx^2} = \frac{d^2u}{dy^2} = \sec\omega\,\frac{d^2u}{dx\,dy}.$$

The equation of two tangents drawn from a point (xy) is

$$4u^2U^2 = \left(X\frac{d(u^2)}{dx} + Y\frac{d(u^2)}{dy} + \ldots\right),$$

and the condition for these being at right angles is, if

$$u^2 \equiv (a, b, c, f, g, h \between x, y, 1)^2,$$
$$4u^2(a+b-2h\cos\omega) = \left(\frac{du^2}{dx}\right)^2 + \left(\frac{du^2}{dy}\right)^2 - 2\left(\frac{du^2}{dx}\right)\left(\frac{du^2}{dy}\right)\cos\omega.$$

But $\quad a = \frac{1}{2}\frac{d^2(u^2)}{dx^2} = \left\{\left(\frac{du}{dx}\right)^2 + u\frac{d^2u}{dx^2}\right\}\,;$

$$b = \left(\frac{du}{dy}\right)^2 + u\frac{d^2u}{dx\,dy},\quad h = \frac{du}{dx}\frac{du}{dy} + u\frac{d^2u}{dx\,dy}\,;$$

so that the equation becomes

$$4u^2\left\{\left(\frac{du}{dx}\right)^2 + u\frac{d^2u}{dx^2} + \left(\frac{du}{dy}\right)^2 + u\frac{d^2u}{dy^2} - 2\cos\omega\left(\frac{du}{dx}\frac{du}{dy} + u\frac{d^2u}{dx\,dy}\right)\right\}$$
$$= 4u^2\left(\frac{du}{dx}\right)^2 + 4u^2\left(\frac{du}{dy}\right)^2 - 8\cos\omega\,u^2\frac{d^2u}{dx\,dy},$$

or $\quad 4u^3\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} - 2\cos\omega\frac{d^2u}{dx\,dy}\right) = 0\,;$

that is, the equation of the director circle is

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} - 2\frac{d^2u}{dx\,dy}\cos\omega = 0.$$

A pair of directrices, the director circle, and the conic, have four common points; hence the equation of a pair of parallel directrices will be

$$\lambda = u\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}-2\frac{d^2u}{dx\,dy}\cos\omega\right),$$

where λ is some constant; and the two values of λ can be most readily determined by making this equation represent two parallel straight lines.

But $$u\frac{du}{dx} = ax+hy+g,\quad \left(\frac{du}{dx}\right)^2+u\frac{d^2u}{dx^2} = a;$$

$$\left(\frac{du}{dy}\right)^2+u\frac{d^2u}{dy^2} = b;\quad \frac{du}{dx}\frac{du}{dy}+u\frac{d^2u}{dx\,dy} = h,$$

or the equation for the pair of directrices is

$$\lambda\,(ax^2+by^2+c+2fy+2gx+2hxy) = u^2(a+b-h\cos\omega)-(ax+hy+g)^2$$
$$-(hx+by+f)^2+2\,(ax+hy+g)(hx+by+f)\cos\omega;$$

the coefficient of x^2 is

$$a\lambda-a\,(a+b-2h\cos\omega)+a^2+h^2-2ah\cos\omega,\ \text{ or }\ a\lambda+h^2-ab.$$

Similarly the coefficient of y^2 is $b\lambda+h^2-ab$; and the coefficient of $2xy$ is

$$\lambda h-h\,(a+b-h\cos\omega)+ah+bh-(ab+h^2)\cos\omega,\ \text{ or }\ \lambda h+(h^2-ab)\cos\omega$$

so that the equation for λ is

$$(a\lambda-P)^2\,(b\lambda-P) = (h\lambda-P\cos\omega)^2,\quad\text{where } P = ab-h^2,$$

or $$\lambda^2-\lambda\,(a+b-2h\cos\omega)+(ab-h^2)\sin^2\omega = 0.$$

Hence, to get the equation of all four directrices, we have to eliminate λ between the equations $\lambda = u\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}-2\frac{d^2u}{dx\,dy}\cos\omega\right)$,

$$\lambda^2-\lambda\,(a+b-2h\cos\omega)+(ab-h^2)\sin^2\omega = 0;$$

where $a = \left(\frac{du}{dx}\right)^2+u\frac{d^2u}{dx^2},\quad b = \left(\frac{du}{dy}\right)^2+u\frac{d^2u}{dy^2},\quad h = \frac{du}{dx}\frac{du}{dy}+u\frac{d^2u}{dx\,dy}.$

Hence $$a+b-2h\cos\omega-\lambda = \left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2-2\frac{du}{dx}\frac{du}{dy}\cos\omega,$$

and the final equation is

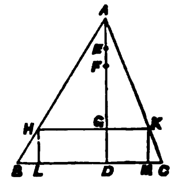
$$u\left\{\left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2-2\frac{du}{dx}\frac{du}{dy}\cos\omega\right\}\left\{\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}-2\frac{d^2u}{dx\,dy}\cos\omega\right\}$$
$$=\left\{\left[\left(\frac{du}{dx}\right)^2+u\frac{d^2u}{dx^2}\right]\left[\left(\frac{du}{dy}\right)^2+u\frac{d^2u}{dy^2}\right]-\left(\frac{du}{dx}\frac{du}{dy}+u\frac{d^2u}{dx\,dy}\right)^2\right\}\sin^2\omega.$$

And this will be found (I hope) to coincide with the given equation, which I must have obtained in some different way, long forgotten.

7290. (By S. Tebay, B.A.)—In a given triangle inscribe a rectangle having one side parallel to the base, and the perimeter equal to given straight line.

Solution by R. Knowles, B.A.; Belle Easton; *and others.*

Let ABC be the triangle, AD perpendicular to BC. Make DE = BC, and let DF be half the given straight line. Find AG a fourth proportional to AE, AF, AD, and through G draw HK parallel to BC, and HL, KM perpendicular to BC. Then LK is the rectangle required.



By similar triangles,

$$AD : BC \text{ (or DE)} = AG : HK \text{ (or FG)};$$

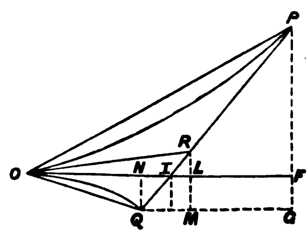
whence $\qquad AE : DE = AF : FG,$

and $\qquad AE : AF = DE : FG = AD : AG,$

which proves the construction.

7205. (By C. Leudesdorf, M.A.)—The tangent at any point P of the cissoid $y^2(a-x) = x^3$ cuts the curve again at Q, and R is a point on PQ such that RP = 2RQ. Show that, if the straight lines joining R, P, Q to the origin make angles θ, α, β respectively with the axis of x, then $\cot\alpha = \tan\alpha - \cot\beta$.

Solution by J. S. Jenkins; Kate Gale; *and others.*



Let (x', y') be the coordinates of P; then the equation to the tangent at P is $y - y' = \dfrac{x'^{\frac{1}{2}}(3a-2x')}{2(a-x')^{\frac{3}{2}}}(x-x')$, therefore $OI = \dfrac{ax'}{3a-2x'}$; and combining the equations of tangent and curve, we find the tangent cuts the curve as represented by

$$(x-x')^2\left[(4a-3x')\,a^2x - a^3x'\right] = 0,$$

whence, for point Q, we have

$$x = \frac{ax'}{4a-3x} = ON, \quad y = \frac{-ax'^{\frac{3}{2}}}{2(a-x')^{\frac{1}{2}}(4a-3x')} = NQ.$$

By similar triangles, $OL = \dfrac{x'(2a-x')}{(4a-3x')}$, $\quad RL = \dfrac{x'^{\frac{3}{2}}(a-x')^{\frac{1}{2}}}{(4a-3x')}$;

whence $\quad \cot\theta = \dfrac{OL}{RL} = \dfrac{2a-x'}{x'^{\frac{1}{2}}(a-x')^{\frac{1}{2}}}$, and $\cot\beta = \dfrac{ON}{NQ} = -\dfrac{2(a-x')^{\frac{1}{2}}}{x'^{\frac{1}{2}}}$,

therefore $\quad \cot\theta + \cot\beta = \dfrac{2a-x'-2(a-x')}{x'^{\frac{1}{2}}(a-x')^{\frac{1}{2}}} = \dfrac{x'^{\frac{1}{2}}}{(a-x')^{\frac{1}{2}}} = \tan\alpha$,

therefore $\qquad \cot\theta = \tan\alpha - \cot\beta.$

6941. (By the Rev. T. W. OPENSHAW, M.A.)—Find the equation to the circle circumscribing the triangle formed by two tangents to the ellipse $a^2y^2+b^2x^2=a^2b^2$ and their chord of contact.

Solution by the PROPOSER; BELLE EASTON; *and others.*

If (h, k) be the point of intersection of the tangents, the equation to a circle through the points of contact will be of the form

$$(a^2yk+b^2xh-a^2b^2)\,(a^2yk-b^2xh-d)=\lambda\,(a^2y^2+b^2x^2-a^2b^2).$$

The conditions for a circle, and that the circle shall pass through (h, k), give

$$\lambda=\frac{a^4k^2+b^4h^2}{a^2-b^2},\quad d=-\frac{a^2b^2(h^2+k^2)}{a^2-b^2}.$$

By substitution, the equation to the circle is

$$(a^2k^2+b^2h^2)(x^2+y^2)-b^2h\,(h^2+k^2+a^2-b^2)\,x-a^2k\,(h^2+k^2-a^2+b^2)\,y$$
$$-(a^2k^2-b^2h^2)(a^2-b^2)=0.$$

If (h, k) is on the ellipse, this will give the osculating circle.

3269. (By the EDITOR.)—Prove that the chord that joins the points $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ on the conic $l\alpha^{\frac12}+m\beta^{\frac12}+n\gamma^{\frac12}=0$ is parallel to

$$\frac{l\alpha}{\alpha_1^{\frac12}+\alpha_2^{\frac12}}+\frac{m\beta}{\beta_1^{\frac12}+\beta_2^{\frac12}}+\frac{n\gamma}{\gamma_1^{\frac12}+\gamma_2^{\frac12}}=0.$$

Solution by the Rev. J. L. KITCHIN, M.A.; N. SARKAR, B.A.; *and others.*

Let $\quad l_1\alpha+m_1\beta+n_1\gamma=0$ be the equation to the chord;

then $\quad l_1\alpha_1+m_1\beta_1+n_1\gamma_1=0,\ \ l_1\alpha_2+m_1\beta_2+n_1\gamma_2=0;$

therefore $\quad \begin{vmatrix}\alpha, & \beta, & \gamma\\ \alpha_1, & \beta_1, & \gamma_1\\ \alpha_2, & \beta_2, & \gamma_2\end{vmatrix}=0$, the equation to the chord,

which is $\quad \alpha\,(\beta_1\gamma_2-\beta_2\gamma_1)+\beta\,(\alpha_2\gamma_1-\alpha_1\gamma_2)+\gamma\,(\alpha_1\beta_2-\alpha_2\beta_1)=0;$

but $\quad l\,\alpha_1^{\frac12}+m\,\beta_1^{\frac12}+n\,\gamma_1^{\frac12}=0,\ \ l\,\alpha_2^{\frac12}+m\,\beta_2^{\frac12}+n\gamma_2^{\frac12}=0;$

therefore $\quad \dfrac{l}{(\beta_1\gamma_2)^{\frac12}-(\beta_2\gamma_1)^{\frac12}}=\dfrac{m}{(\alpha_2\gamma_1)^{\frac12}-(\alpha_1\gamma_2)^{\frac12}}=\dfrac{n}{(\alpha_1\beta_2)^{\frac12}-(\alpha_2\beta_1)^{\frac12}};$

$\therefore\ \alpha l\,[(\beta_1\gamma_2)^{\frac12}+(\beta_2\gamma_1)]^{\frac12}+m\beta\,[(\alpha_2\gamma_1)^{\frac12}+(\alpha_1\gamma_2)]^{\frac12}+n\gamma\,[(\alpha_1\beta_2)^{\frac12}+(\alpha_2\beta_1)]=0.$

Now, if this line move parallel to itself up to $(\alpha', \beta', \gamma')$, it becomes

$$2\,\alpha l\,(\beta'\gamma')^{\frac12}+2m\beta\,(\alpha'\gamma')^{\frac12}+2n\gamma\,(\alpha'\beta')^{\frac12}=0,$$

or $\quad \dfrac{l\alpha}{\alpha'^{\frac12}}+\dfrac{m\beta}{\beta'^{\frac12}}+\dfrac{n\gamma}{\gamma'^{\frac12}}=0$, the tangent.

But if the line given in the question move in the same way, we get

$$\frac{l\alpha}{2\alpha'^{\frac12}}+\frac{m\beta}{2\beta'^{\frac12}}+\frac{n\gamma}{2\gamma'^{\frac12}}=0,\quad \text{or}\quad \frac{l\alpha}{\alpha'^{\frac12}}+\frac{m\beta}{\beta'^{\frac12}}+\frac{n\gamma}{\gamma'^{\frac12}}=0.$$

Hence both lines are parallel to the tangent, and are therefore parallel.

7201. (By R. F. Scott, M.A.)—Prove that

$$\int_0^\pi x \log(1-\sin^2 a \sin^2 x)\, dx = 2\pi^2 \log\cos\tfrac{1}{2}a.$$

Solution by D. Edwardes; Prof. Matz, M.A.; *and others.*

Writing $\pi - x$ for x, the integral is $\tfrac{1}{2}\pi \int_0^\pi \log(1-\sin^2 a\sin^2 x)\, dx.$

If
$$u = \int_0^\pi \log(1-\sin^2 a \sin^2 x)\, dx,$$

$$\frac{du}{da} = -\sin 2a \int_0^\pi \frac{\sin^2 x}{1-\sin^2 a\sin^2 x}\, dx = -2\pi\tan\tfrac{1}{2}a;$$

therefore $u = 4\pi\log\cos\tfrac{1}{2}a + C$. But, when $a=0$, $u=0$, therefore $C=0$; hence $u = 4\pi\log\cos\tfrac{1}{2}a$; therefore $I = \tfrac{1}{2}\pi u = 2\pi^2\log\cos\tfrac{1}{2}a$.

7072. (By Ath Bigah Bhut.)—If a^3, b^3, c^3, d^3 denote

$$\begin{vmatrix} x, & y, & z \\ u, & x, & y \\ z, & u, & x \end{vmatrix}, \quad \begin{vmatrix} y, & z, & u \\ x, & y, & z \\ u, & x, & y \end{vmatrix}, \quad \begin{vmatrix} z, & u, & x \\ y, & z, & u \\ x, & y, & z \end{vmatrix}, \quad \begin{vmatrix} u, & x, & y \\ z, & u, & x \\ y, & z, & u \end{vmatrix};$$

exhibit the values (severally) of x, y, z, u, in terms of a, b, c, d.

Solution by W. H. Blythe, M.A.; Belle Easton; *and others.*

The following are evident identities, reducing to determinants with two identical rows

$$d^3x + a^3y + b^3y + c^3u = 0, \quad c^3x + d^3y + a^3y + b^3u = 0, \quad b^3x + c^3y + d^3y + a^3u = 0;$$

hence
$$\frac{x}{\begin{vmatrix} a^3, & b^3, & c^3 \\ d^3, & a^3, & b^3 \\ c^3, & d^3, & a^3 \end{vmatrix}} = \frac{y}{\begin{vmatrix} b^3, & c^3, & d^3 \\ a^3, & b^3, & c^3 \\ d^3, & a^3, & b^3 \end{vmatrix}}$$
= two similar expressions in z, and u, a^3, b^3, c^3, d^3.

If we write this $\dfrac{x}{\lambda_1} = \dfrac{y}{\lambda_2} = \dfrac{z}{\lambda_3} = \dfrac{u}{\lambda_4} = \mu$, $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ being known in terms of a, b, c, d, substitute $x = \lambda_1\mu$, &c., for x, y, z, u in the first determinant, we obtain

$$a^3 = \mu^3 \begin{vmatrix} \lambda_1, & \lambda_2, & \lambda_3 \\ \lambda_4, & \lambda_1, & \lambda_2 \\ \lambda_3, & \lambda_4, & \lambda_1 \end{vmatrix}, \quad x = a \begin{vmatrix} a^3, & b^3, & c^3 \\ d^3, & a^3, & b^3 \\ c^3, & d^3, & a^3 \end{vmatrix} \div \begin{vmatrix} \lambda_1, & \lambda_2, & \lambda_3 \\ \lambda_4, & \lambda_1, & \lambda_2 \\ \lambda_3, & \lambda_4, & \lambda_1 \end{vmatrix}^{\frac{1}{3}},$$

where
$$\lambda_1 = \begin{vmatrix} a^3, & b^3, & c^3 \\ d^3, & a^3, & b^3 \\ c^3, & d^3, & a^3 \end{vmatrix}$$

with similar expression for λ_2, λ_3, and λ_4.

6832. (By Professor Matz, M.A.)—Find the values of X and x from the equations $\quad \sin X^{\log x} = \dfrac{1}{\pi}, \quad \log (Xx) = \dfrac{2}{\pi}.$

Solution by G. M. Reeves, M.A.; J. O'Regan; *and others.*

Let $u = \log X$ and $v = \log x$; then the equations are equivalent to

$$uv = \log \sin^{-1} \frac{1}{\pi}, \quad u + v = \frac{2}{\pi}; \text{ whence we obtain}$$

$$u = \frac{1}{\pi} \pm \left(\frac{1}{\pi^2} - \log \sin^{-1} \frac{1}{\pi} \right)^{\frac{1}{2}} = \log X,$$

$$v = \frac{1}{\pi} \mp \left(\frac{1}{\pi^2} - \log \sin^{-1} \frac{1}{\pi} \right)^{\frac{1}{2}} = \log x.$$

7138. (By G. G. Morrice, B.A.)—A triangle Δ is formed by the straight lines $a_1x + b_1y = c_1$, $a_2x + b_2y = c_2$, $a_3x + b_3y = c_3$; another triangle Δ_1 is formed by the external bisectors of its angles Δ_2 by the external bisectors of Δ_1; show that, if

$$s_1 = a_1 + a_2 + a_3, \quad s_2 = b_1 + b_2 + b_3, \quad s_3 = c_1 + c_2 + c_3,$$

Area of Δ_{2r} is

$$\frac{\tfrac{1}{2} \left| \tfrac{1}{3}(4^r - 1)s_1 + a_1, \ \tfrac{1}{3}(4^r - 1)s_2 + b_2, \ \tfrac{1}{3}(4^r - 1)s_3 + c_3 \right|^2}{\left| \tfrac{1}{3}(4^r-1)s_1 + a_1, \ \tfrac{1}{3}(4^r-1)s_2 + b_2 \right| \times \left| \tfrac{1}{3}(4^r-1)s_1 + a_2, \ \tfrac{1}{3}(4^r-1)s_2 + b_3 \right| \times \left| \tfrac{1}{3}(4^r-1)s_1 + a_3, \ \tfrac{1}{3}(4^r-1)s_2 + b_1 \right|}.$$

Solution by the Proposer; Prof. Nash, M.A.; *and others.*

$$\text{The area of } \Delta_0 = \frac{\tfrac{1}{2} \begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}^2}{\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix} \times \begin{vmatrix} a_2, & b_2 \\ a_3, & b_3 \end{vmatrix} \times \begin{vmatrix} a_3, & b_3 \\ a_1, & b_1 \end{vmatrix}}.$$

The equation of the external bisector of the angle between $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ is $(a_1 + a_2)x + (b_1 + b_2)y = c_1 + c_2$.

$$\text{Hence} \qquad \Delta_1 = \frac{\tfrac{1}{2} \begin{vmatrix} s_1 - a_1, & s_2 - b_1, & s_3 - c_1 \\ s_1 - a_2, & s_2 - b_2, & s_3 - c_2 \\ s_1 - a_3, & s_2 - b_3, & s_3 - c_3 \end{vmatrix}^2}{\begin{vmatrix} s_1 - a_1, & s_2 - b_1 \\ \ldots & \ldots \quad \ldots \\ \ldots & \ldots \quad \ldots \end{vmatrix} \times \begin{vmatrix} \ldots \\ \ldots \\ \ldots \end{vmatrix} \times \begin{vmatrix} \ldots \\ \ldots \\ \ldots \end{vmatrix}}.$$

Proceeding in the same way by substituting for any element the sum of the remaining elements in the same column, we have successively corresponding to a_1 the elements $s_1 + a_1$, $3s_1 - a_1$, $5s_1 + a_1$, $11s_1 - a_1$, $21s_1 + c_1$,

$43s_1-a_1$, $85s_1+a_1$. Now the series $1+3+5+11+21+43+\dots$ may be written $1+(2^1+1)+(2^2+1)+(2^3+3)+(2^4+5)+\dots$ So that any term $T_r = T_{r-2}+2^r$, $T_{r-2} = T_{r-4}+2^{r-2}$, $\therefore$ $T_r = T_0+2^r+2^{r-2}+2^2 = \dfrac{4^r-1}{3}$.

$$\text{Hence}\qquad \Delta_{2r} = \frac{\frac{1}{2}\begin{vmatrix} \frac{4^r-1}{3}s_1+a_1, & \cdots \\ & \cdots \\ \cdots \quad \cdots & \cdots \end{vmatrix}^2}{\begin{vmatrix} \cdots\cdots \\ \cdots\cdots \end{vmatrix} \times \begin{vmatrix} \cdots\cdots \\ \cdots\cdots \end{vmatrix} \times \begin{vmatrix} \cdots\cdots \\ \cdots\cdots \end{vmatrix}}.$$

6953 & 7126. (By Professor Wolstenholme, M.A., D.Sc.)—(6953.) A circle is drawn with its centre O on the parabola $y^2=4ax$, and such that triangles can be inscribed in the parabola whose sides touch the circle: prove that (1) the radius of the circle is twice the normal to the parabola at O cut off by the axis; (2) the envelope of these circles consists of two distinct curves, one of which is the parabola $y^2 = 4a(x+4a)$, and the other is a quartic of the fourth class, whose equation may be written

$$2y^2+x^2-38ax-239a^2 = (x+21a)^{\frac{3}{2}}(x+5a)^{\frac{1}{2}};$$

(3) if the circle touch these curves in the points P, Q, the tangents at O, P, Q to their respective loci concur in a point which is the polar with respect to the parabola of the normal at O; and (4) if θ be the angle which the tangent at O makes with the axis, the tangents at P, Q will make angles $-\theta$, 3θ (or $3\theta\pm\pi$) with the axis. (The quartic envelope is the first negative pedal of the curve whose equation referred to the focus as pole is $r = 3a\sec\frac{1}{3}\theta$.)

(7126.) With a point O on the ellipse $a^2y^2+b^2x^2 = a^2b^2$ as centre is described a circle such that triangles can be circumscribed to the circle and inscribed in the ellipse; prove that (1) the envelope of such circles consists of two distinct curves, of which one is the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = \left(\dfrac{a^2+b^2}{a^2-b^2}\right)^2$, and the other is a curve of order 6, class 6, having 6 nodes (2 or 4 real), 6 bitangents (6 or 4 real), 4 cusps, and 4 inflexions (probably all impossible), so that its reciprocal has the same Plückerian numbers, and osculating the elliptic envelope in four points (where the normal and tangent are equally inclined to the axes). Also (2), if P, Q be the points of contact of the circle with these two curves, the tangents at O, P, Q will concur in one point, which is the polar with respect to the given ellipse of the normal at O; and if θ be the angle which the tangent at O makes with the axis, the tangents at P, Q will make angles $\pi-\theta$, 3θ with the axis. If $a^2>3b^2$, the maximum distance of Q from the centre is $(a^2+b^2)^{\frac{3}{2}}\div(a^2-b^2)$; (3) the radius of curvature at Q of the locus of Q is $\dfrac{a^2b^2}{p^3}\cdot\dfrac{4p^2+a^2+b^2}{3(a^2-b^2)}$, where p is the perpendicular from the centre on the tangent at O; and (4) trace the locus of Q when $b^2 = 2a^2$, $3a^2$, $4a^2$, $-2a^2$, $-3a^2$, $-5a^2$.

Solution by the Proposer.

(6953.) If we take the coordinates of O to be am^2, $2am$, then $m = \cot\theta$, and the equation of a circle with centre O is $(x-am^2)^2+(y-2am)^2 = r^2$;

and forming the discriminant of $k(y^2-4ax)+(x-am^2)^2+(y-2am)^2-r^2$, we find $\Delta = 4a^2,\ \Theta = 4a^2(1+m^2),\ \Theta' = r^2,\ \Delta' = r^2$,
and the condition that triangles can be inscribed in the parabola whose sides touch the circle is $\Theta'^2 = 4\Theta\Delta'$, *i.e.*, $r^2 = 0$ or $r^2 = 16a^2(1+m^2)$. For real triangles the second is the only available value, and we see that, if OG be the normal at O terminated by the axis, the radius of the circle will be 2OG. But, if P be a point on the parabola $y^2 = 4a(x+4a)$ such that $x+4a = am^2$, $y = -2am$, and OM, PN be perpendicular to the axis; then, if OP meet the axis in T, we have $\text{MN} = 4a$, $\text{NT} = 2a$, or PO is the normal at P. Thus the parabola $y^2 = 4a(x+4a)$ touches every such circle and is part of the envelope. The other point of contact of the circle with its envelope will then be found by drawing PQ perpendicular to the tangent at O, and bisected by this tangent (for this tangent is the line joining consecutive centres of circles). We get then, for the coordinates of Q,

$$x+4a-am^2 = \frac{y+2am}{-m},\quad x-4a+am^2-m(y-2am)+2am^2 = 0;$$

whence $x+5a = \dfrac{a(3-m^2)^2}{1+m^2}$, $y = 2am\dfrac{3m^2-5}{1+m^2}$, also $x+21a = \dfrac{a(5+m^2)^2}{1+m^2}$,
from which equations may be obtained

$$2y^2+x^2-38ax-239a^2 = (x+21a)^{\frac{3}{2}}(x+5a)^{\frac{1}{2}} \ldots\ldots\ldots\ldots\ldots (2).$$

The tangents to the circle at P, Q will intersect on the tangent at O, since this is the join of two consecutive centres; whence (3) the tangents at O, P, Q concur in one point, and this point (U) is given by the equations $x-my+am^2 = 0,\quad x+4a+my+am^2 = 0$,
that is, $x+2a+am^2 = 0,\quad my+2a = 0$,
whence the locus of U is $y^2(x+2a)+4a^3 = 0$. Also, (4) the tangents at O, P being equally inclined to the axis, that at P makes an angle $\pi-\theta$ with the axis, and the angle $\text{UQP} = \text{UPQ} = \text{POU} = \frac{1}{2}\pi-2\theta$, whence the angle which UQ, the tangent at Q, makes with the axis is $\frac{1}{2}\pi+\theta-(\frac{1}{2}\pi-2\theta)$ or 3θ. (Here θ has been assumed $<\frac{1}{4}\pi$; the angle may be 3θ or $3\theta\pm\pi$.)

The equation of the tangent at Q will then be

$$y+2a\tan\theta = \tan 3\theta\,(x+2a+a\cot^2\theta);$$

and the perpendicular on it from S, the focus of the given parabola, will be $a(3+\cot^2\theta)\sin 3\theta-2a\tan\theta\cos 3\theta$

$$\equiv \frac{2a}{\cos\theta}(\sin 3\theta\cos\theta-\cos 3\theta\sin\theta)+\frac{a\sin 3\theta}{\sin^2\theta} \equiv 4a\sin\theta+\frac{a\sin 3\theta}{\sin^2\theta} \equiv \frac{3a}{\sin\theta}.$$

Hence the equation of the pedal of the locus of Q with respect to S is $r = 3a\sec\frac{1}{3}\theta$; or the locus of Q is the first negative pedal of this curve, when S is pole.

[The quartic has two cusps (impossible) when $m^2+5 = 0$, $x+21a = 0$, $y^2+500a^2 = 0$; one crunode when $m^2 = \frac{5}{3}$, $y = 0$, $x = -\frac{13}{3}a$; one bitangent when $m^2 = 3$, $x = -5a$, $y^2 = 12a^2$. Plücker's equations then prove that the class of the curve is 4, and that there are two inflexions which I suppose are impossible from tracing the curve. The curve and its reciprocal will therefore have all Plücker's numbers the same for the two curves ($m = n = 4$, $\delta = \tau = 1$, $\kappa = \iota = 2$). It would be interesting to receive an investigation as to whether any projection of this curve could be its own reciprocal.

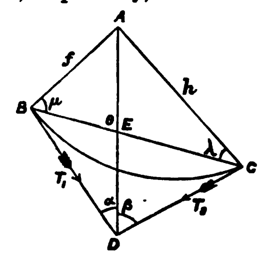
(7126.) This is the corresponding Question for the *ellipse*, and an analogous investigation would prove the theorems in this Question.

7321. (By D. EDWARDES.)—The extremities of a heavy uniform string are attached to the ends of a weightless bent lever, whose arms are at right angles to one another and of lengths f, h. If α, β, θ are the inclinations to the vertical, in the position of equilibrium, of the tangents to the string at its extremities and of the line joining its extremities, prove that

$$\cot\theta = \frac{f^2\cot\alpha - h^2\cot\beta}{h^2+f^2-hf(\cot\alpha+\cot\beta)}.$$

Solution by DR. CURTIS; W. H. BLYTHE, M.A.; *and others.*

If λ, μ, denote the angles opposite to f and h, respectively, in the triangle, whose vertex is at the fulcrum A, and whose sides are f and h, or AB and AC, while T_1, T_2 denote the tensions along BD, CD, the tangents at the extremities of the catenary in which the string hangs; then, for the equilibrium of the lever,



$$T_1\sin\alpha - T_2\sin\beta = 0,$$

and, taking moments round A,

$$T_1 f\sin(\theta-\alpha+\mu) = T_2 h\sin(\theta+\beta-\lambda),$$

or

$$f\sin\beta\sin(\theta-\alpha+\mu) = h\sin\alpha\sin(\theta+\beta-\lambda);$$

expanding $\sin(\theta-\alpha+\mu)$ and $\sin(\theta+\beta-\lambda)$, and remembering that

$$\sin\lambda = \cos\mu = \frac{f}{\sqrt{(f^2+h^2)}},\quad \cos\lambda = \sin\mu = \frac{h}{\sqrt{(f^2+h)^2}},$$

we easily obtain the result,

$$\cot\theta = \frac{f^2\cot\alpha - h^2\cot\beta}{f^2+h^2-fh(\cot\alpha+\cot\beta)} \quad\ldots\ldots\ldots\ldots (1),$$

It is plain, however, that this equation is insufficient to solve the problem, which involves *three* unknown quantities, α, β, and θ. Two other equations must consequently be obtained. Now, as the weight of the lever is neglected, it is necessary for equilibrium that the vertical through D, the line along which the weight of the catenary acts, should pass through the fulcrum A, therefore (*geometrically*)

$$\frac{f\sin(\theta+\mu)}{h\sin(\theta-\lambda)} = \frac{\mathrm{BE}}{\mathrm{CE}} = \frac{\sin(\theta+\beta)\sin\alpha}{\sin(\theta-\alpha)\sin\beta},$$

or

$$\frac{\sin(\theta+\beta)\sin\alpha}{\sin(\theta-\alpha)\sin\beta} = \frac{f}{h}\cot(\theta-\lambda) \quad\ldots\ldots\ldots\ldots (2).$$

Again, if k be the parameter of the catenary, and y_1, y_2 the vertical ordinates of B and C,

$$\sqrt{(f^2+h^2)}\sin\theta = y_2 - y_1 = k\left(\frac{1}{\sin\beta} - \frac{1}{\sin\alpha}\right) = k\,\frac{\sin\alpha-\sin\beta}{\sin\alpha\sin\beta},$$

while l, the given length of the string,

$$= k(\cot\alpha+\cot\beta) = k\,\frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta},$$

therefore

$$\frac{\sqrt{(f^2+h^2)}\sin\theta}{l} = \frac{\sin\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}(\alpha+\beta)} \quad\ldots\ldots\ldots\ldots (3).$$

Equations (1), (2), and (3) solve the problem. It is not difficult to discuss in a similar manner the question in which the weight of the lever (homogeneous or not) is considered, and the arms are inclined at any angle.

6925. (By Professor MATZ, M.A.)—Solve the equation

$$2\left[\log\left(1+\sin^2\theta\right)\right]^{\frac{1}{2}} = \frac{a\left[2-\log\left(2-\cos^2\theta\right)\right]}{\left[1-\log\left(2-\cos^2\theta\right)\right]^{\frac{1}{2}}}.$$

Solution by J. HAMMOND, M.A.; Professor COCHEZ; *and others.*

Writing $2-\log\left(2-\cos^2\theta\right) = 2x$, the equation is

$$(2-2x)^{\frac{1}{2}} = \frac{ax}{(2x-1)^{\frac{1}{2}}}, \quad \text{or} \quad a^4x^4-8x^3+20x^2-16x+4 = 0.$$

The completion of the square gives $(x^2-4x+2)^2 = (1-a^4)\,x^4$.........(1),

whence $x^2\left[1-(1-a^4)^{\frac{1}{2}}\right]-4x+2 = 0$, or $x = \dfrac{2\pm\left[2+2\,(1-a^4)^{\frac{1}{2}}\right]^{\frac{1}{2}}}{1-(1-a^4)^{\frac{1}{2}}}$.

But $\quad 2+2\,(1-a^4)^{\frac{1}{2}} = \left[(1+a^2)^{\frac{1}{2}}+(1-a^2)^{\frac{1}{2}}\right]^2 = \alpha^2$, suppose.

Then $\quad x = \dfrac{2\,(2\pm\alpha)}{4-\alpha^2} = \dfrac{2}{2\pm\alpha} = \dfrac{2}{2\pm(1+a^2)^{\frac{1}{2}}\pm(1-a^2)^{\frac{1}{2}}}$.

Neglecting the ambiguities, which can be introduced again at pleasure,

$$x = \frac{2}{2+\alpha}, \quad \log\left(2-\cos^2\theta\right) = \frac{2\alpha}{2+\alpha}, \quad 2-\cos^2\theta = e^{\frac{2\alpha}{2+\alpha}}.$$

The simplest form of the final result is $\sin\theta = \left(e^{\frac{2\alpha}{2+\alpha}}-1\right)^{\frac{1}{2}}$.

7395. (By R. TUCKER, M.A.)—If we have

$$\Delta \equiv \begin{vmatrix} ac^2, & ba^2, & cb^2 \\ ab^2, & bc^2, & ca^2 \\ \cos A, & \cos B, & \cos C \end{vmatrix} \text{ and } \Delta' \equiv \begin{vmatrix} ac, & a^2, & bc \\ ab, & bc, & a^2 \\ \frac{1}{2}, & \cos B, & \cos C \end{vmatrix},$$

where the elements involved are those of a plane triangle, prove that

$$2\Delta = (a^2+b^2+c^2)\,\Delta'.$$

Solution by G. G. MORRICE, B.A.; J. O'REGAN; *and others.*

Putting, in the last row of Δ, $\cos A = \dfrac{a\,(b^2+c^2-a^2)}{2abc}$, with two similar substitutions, and then subtracting the first two elements in each column

from the third, we get

$$-\Delta = \begin{vmatrix} c^2, & b^2, & a^2 \\ a^2, & c^2, & b^2 \\ b^2, & a^2, & c^2 \end{vmatrix} = (a^2+b^2+c^2)\begin{vmatrix} 1, & b^2, & a^2 \\ 1, & c^2, & b^2 \\ 1, & a^2, & c^2 \end{vmatrix}$$

$$= (a^2+b^2+c^2)(a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2);$$

$$2\Delta' = a^4-b^2c^2+2a^3(b\cos C+c\cos B)+2abc(b\cos B+c\cos C);$$

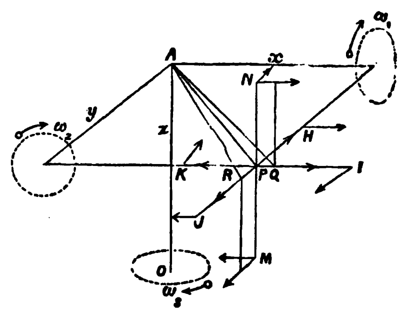
but $2abc\,.\,b\cos B = b^2(c^2+a^2-b^2)$, $2abc\,.\,c\cos C = c^2(a^2+b^2-c^2)$;

therefore $2\Delta' = a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2$, therefore, &c.

7403. (By Professor Sylvester, F.R.S.)—From the principle of conservation of areas, deduce geometrically Euler's equations for the motion of a body revolving about a fixed point.

I. *Solution by* G. S. Carr, M.A.

Let P be an element of the body, O the fixed point and origin of coordinates, ω_1, ω_2, ω_3 the resolved angular velocities.



The impressed couple about the z axis of rotation is proportional to the acceleration in the description of areas about that axis by the point P. ω_3 contributes directly to this the area

$$\tfrac{1}{2}\mathrm{AP}^2\frac{d\omega_3}{dt} = \tfrac{1}{2}(x^2+y^2)\frac{d\omega_3}{dt}$$

(not shown in the figure).

Let the velocity of P parallel to z and due to ω_1 be represented by PN $= y\omega_1$, and that due to ω_2 by PM $= x\omega_2$. The acceleration parallel to x, caused by PN and ω_2, is $y\omega_1\omega^2$ at N, which may be superposed at P and represented by PQ. Similarly the acceleration parallel to y, caused by PM and ω_1, is $x\omega_2\omega_1$ at M, represented by PR. The areas due to these accelerations are $+$APR and $-$APQ, that is, $\frac{1}{2}x^2\omega_2\omega_1$ and $-\frac{1}{2}y^2\omega_1\omega_2$.

Thus for the whole acceleration of the momentum round the z axis (referred to principal axes of the body), we have

$$\Sigma(x^2+y^2)\frac{d\omega_3}{dt}-\Sigma(y^2-x^2)\,\omega_1\omega_2, \quad \text{or} \quad C\frac{d\omega_3}{dt}-(A-B)\,\omega_1\omega_2 = N,$$

the impressed couple; and this is Euler's third equation.

There are eight areas in addition to the foregoing due to accelerations indicated by the arrows. But two of these arising from accelerations at J and K are always equal and of opposite sign, while the rest disappear in the summation when principal axes are taken.

The entire twelve areas are as follows. The first factor of each is the perpendicular from A upon the base of the triangle, which base represents an acceleration of velocity. Four areas due to direct accelerations at P, parallel to the x and y axes (the double area is written in each case) :—

$$xx\frac{d\omega_3}{dt},\quad yy\frac{d\omega_3}{dt},\quad -yz\frac{d\omega_2}{dt},\quad -xz\frac{d\omega_1}{dt}.$$

And eight areas due to accelerations at right angles to velocities, viz. :—

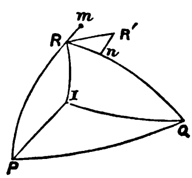
$$\text{at M, } yx\omega_2^2 \text{ and } xx\omega_2\omega_1;\quad \text{at N, } -yy\omega_1\omega_2 \text{ and } -xy\omega_1^2;$$

$$\text{at I, } xz\omega_2\omega_3;\quad \text{at J, } yx\omega_3^2;\quad \text{at K, } -xy\omega_3^2;\quad \text{at H, } -yz\omega_1\omega_3.$$

Putting A, B, C, F, G, H for the moments and products of inertia about the axes (when not principal ones), these areas give the known result

$$-G\frac{d\omega_1}{dt}-F\frac{d\omega_2}{dt}+C\frac{d\omega_3}{dt}-(A-B)\,\omega_1\omega_2-H\,(\omega_1^2-\omega_2^2)-F\omega_3\omega_1+G\omega_2\omega_3=N.$$

II. *Solution by the* Proposer.



1. Suppose no forces acting on the body, then there will be an invariable line perpendicular to the resultant instantaneous axes. Of a sphere described about the fixed point, let P, Q, R be the intersections with the principal axes, and I with the invariable line at the time t.

Let $\text{IP}=\lambda$, $\text{IQ}=\mu$, $\text{IR}=\nu$, and let p, q, r be the angular velocities, and A, B, C the moments of inertia in respect to P, Q, R. Take $\text{R}m = qdt$, $\text{R}n = pdt$, and in the time dt let R get to R′ and call $\text{IR}'=\nu'$; then

$$\frac{Ap}{\cos\lambda}=\frac{Bq}{\cos\mu}=\frac{Cr}{\cos\nu}\quad\text{and}\quad\frac{Cr'}{\cos\nu'}=\frac{Cr}{\cos\nu}=G,$$

the constant angular momentum; hence we have

$$\delta r = 0-\frac{r\sin\nu}{\cos\nu}\,\delta\nu=-\frac{r\sin\nu}{\cos\nu}\,(\text{R}m\cos\text{IRP}-\text{R}n\cos\text{IRQ})$$

$$=-dt\,\frac{r\sin\nu}{\cos\nu}\left(q\,\frac{\cos\lambda}{\sin\nu}-p\,\frac{\cos\mu}{\sin\nu}\right)=dt\left(p\,\frac{Bq}{C}-q\,\frac{Ap}{C}\right).$$

Thus we have $\qquad Cdr=(B-A)\,pqdt;$

and, similarly, $\quad Bdq=(A-C)\,rpdt,\quad Adp=(C-B)\,qrdt.$

2. When external forces are expressed whose moments about P, Q, R are L, M, N, the angular motions of P, Q, R in the time dt will be those due to the motion of the body in the preceding instant of time acted on by no forces during the time dt, *plus* those due to the actions of the forces in the time dt on the body at rest before they begin to act, and consequently the equations become

$$Cdr=(B-A)\,pqdt+N,\quad Bdq=(A-C)\,rpdt+M,\quad Adp=(C-B)\,qrdt+L.$$

6897. (By Professor Townsend, F.R.S.)—An equiangular spiral or spherical surface being supposed the frictionless catenary of a uniform cord, or the frictionless trajectory of a material particle, constrained to

rest or move on the surface, under the action of a force passing perpendicularly in every position through the axis of the spiral; show that the force varies, for the catenary inversely as the square, and for the trajectory inversely as the cube, of the distance from the axis.

Solution by the PROPOSER.

Denoting by α the constant angle of intersection of the spiral with all the meridians of the surface that pass through its poles, by T the current tension of maintenance at any point of the catenary, by V the current velocity of description at any point of the trajectory, by ρ the current distance of the point from the axis of the spiral, and by F the current force per unit of length or of mass at the point; then since, on elementary dynamical principles, for the catenary $d\mathrm{T} = -\mathrm{F}\,d\rho$ and $\mathrm{T}\rho \sin\alpha = \text{const.} = h$, and for the trajectory $\mathrm{V}d\mathrm{V} = \mathrm{F}d\rho$ and $\mathrm{V}\rho\sin\alpha = \text{const.} = k$, therefore at once $\mathrm{F} = h \operatorname{cosec}\alpha \,.\, \rho^{-2}$ in the former case, and $= -k^2 \operatorname{cosec}^2\alpha\,.\,\rho^{-3}$ in the latter case; and therefore, &c.

Since manifestly, as in the corresponding well-known cases of equilibrium and of motion in a plane, the tension of maintenance T in the former case is that to infinity, and the velocity of description V in the latter case is that from infinity, under the action of the force; hence, conversely, a uniform cord or material particle, constrained to rest or move without friction on a spherical surface with the tension of maintenance or the velocity of description to or from infinity, under the action of a repulsive force varying inversely as the square, or of an attractive force varying inversely as the cube of the distance from a fixed diameter of the sphere, through which it passes perpendicularly in every position, will have for its catenary or trajectory an equiangular spiral of the surface of which the diameter is the axis.

Denoting by μ the absolute force of the repulsion or attraction, it appears at once from the above that $\mu = h \operatorname{cosec}\alpha$ for the catenary, and $= k^2 \operatorname{cosec}^2\alpha$ for the trajectory. Hence, for different equiangular spirals about the same axis of force, whether belonging or not to the same spherical surface having its centre on the axis, the constants h and k, as for the corresponding cases of equilibrium and motion in a plane, vary as $\sin\alpha$ in both cases alike.

7036. (By R. TUCKER, M.A.)—Find (1) the maximum triangle, inscribed in an ellipse, two of whose sides pass through the foci; and show (2) that in this case when the excentricity equals $\frac{1}{3}\sqrt{5}$, the angle between the focal chord is 60°.

Solution by J. S. JENKINS, M.A.; SARAH MARKS; *and others.*

Let ABC be the triangle, the two sides of which pass through the foci, and let the coordinates of C be (x', y'); then, since the sides pass through the foci, we readily obtain the following equations:—

$$\mathrm{AC} = \frac{2\,(a+\epsilon x')^2}{(a+a\epsilon^2)^2 + 2\epsilon x'},\quad \mathrm{BC} = \frac{2\,(a-\epsilon x')^2}{(a+a\epsilon^2)^2 - 2\epsilon x'},\quad \sin \mathrm{ACB} = \frac{2b\epsilon\,(a^2-x'^2)^{\frac{1}{2}}}{a^2-\epsilon^2 x'^2};$$

whence Area $\Delta ABC = \dfrac{4b\epsilon\,(a^2-x'^2)^{\frac{1}{2}}\,(a^2-\epsilon^2x'^2)}{(a+a\epsilon^2)^2-4\epsilon x'^2}$, and by differentiating we find this expression to be a maximum when $x'^2 = \dfrac{3a^2\epsilon^4+6a^2\epsilon^2-a^2}{8\epsilon^2}$, therefore the maximum area of the triangle ABC is $\dfrac{3b^2}{2\sqrt{2}\,.\,a}\,(4a^2-3b^2)^{\frac{1}{2}}$.

Again, $\sin ACB = \dfrac{2b\epsilon\,(a^2-x'^2)^{\frac{1}{2}}}{a^2-\epsilon^2x'^2} = \dfrac{\sqrt{3}}{2}$ (when $\epsilon = \frac{1}{3}\sqrt{5}$) $= \sin 60°$.

7411. (By C. Leudesdorf, M.A.)—S is the focus, A the vertex, of the parabola $y^2 = 4ax$. A conic has double contact with the parabola and also with the circle on SA as diameter; prove that its director circle will envelope the curve $y^2(16x+25a) = 4\,(x+a)\,(a^2+4ax-4x^2)$.

Solution by D. Edwardes; R. Knowles, B.A.; *and others.*

The equation of a conic having double contact with the curves in question will be $\mu^2x^2-2\mu\,(2y^2+x^2-5ax)+(x+3a)^2 = 0$. Writing down the equation of a pair of tangents from x, y: equating to zero the sum of the coefficients of x^2 and y^2, the equation of the director circle becomes

$$(\mu-1)^2\,(x^2+y^2)+(\mu-1)\,(10ax+4a^2)+16ax+25a^2 = 0,$$

whose envelope is $a\,(5x+2a)^2 = (x^2+y^2)\,(16x+25a)$, which reduces to the stated form.

7409. (By W. S. M'Cay, M.A.)—Two circles A, B are inverted from an origin O into two circles A′, B′; if O be on a polar with respect to A or B of either of their centres of similitude, prove that after inversion O will still be on a polar with respect to B′ or A′ of one of their centres of similitude.

Solution by G. B. Mathews, B.A.; Kate Gale; *and others.*

Let the circles be $(x-d)^2+y^2 = r^2,\ (x-d')^2+y^2 = r'^2$............(A, B). The polar of the origin with regard to (A) is $dx = d^2-r^2$; and if this goes through the centre of similitude

$$d\,.\,\frac{r'd+rd'}{r+r'} = d^2-r^2,\quad rdd' = rd^2-(r+r')\,r^2;$$

therefore $\quad r\,(r+r') = d\,(d-d')$ or $d^2-r^2 = rr'+dd'$....................(1).

Now, if $\delta\rho$, $\delta'\rho'$ refer to the inverse circles, it is easily seen, geometrically, that $\quad \delta = \frac{1}{2}\left(\dfrac{k^2}{d-r}+\dfrac{k^2}{d+r}\right) = \dfrac{k^2d}{d^2-r^2},\ \rho = \frac{1}{2}\left(\dfrac{k^2}{d-r}-\dfrac{k^2}{d+r}\right) = \dfrac{k^2r}{d^2-r^2}$;

and similarly, $\delta' = \dfrac{k^2 d'}{d'^2 - r'^2}$, $\rho' = \dfrac{k^2 r'}{d'^2 - r'^2}$;

therefore $\rho\rho' + \delta\delta' = \dfrac{k^4 (rr' + dd')}{(d^2 - r^2)(d'^2 - r'^2)} = \dfrac{k^4}{d'^2 - r'^2}$ by (1) $= \delta'^2 - \rho'^2$,

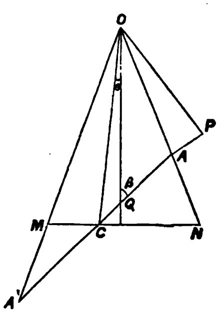
an equation of the same form as (1) with accented letters for unaccented, and *vice versâ:* hence the theorem.

[The Proposer remarks that the theorem *must* be true if A, B are orthogonal, for the tangents from O are then harmonic (Townsend's *Modern Geometry*, Vol. I, p. 287, and *Educational Times* for June, 1878), and, being unchanged by inversion, O must also be on the locus of points from which tangents to A′, B′ are harmonic. In looking for a geometric proof (which is not difficult) of the theorem, it turned out to be true for any two circles. It appears also, more generally, that for any position of O, the figure formed by O, the two centres of similitude S, S′ of A, B, and the two inverses of S, S′ to A and B, preserves its species (but in opposite cyclical order) after inversion.]

7412. (By J. J. Walker, M.A., F.R.S.)—The sides of a right cone make an angle α with the axis; prove that the locus of centres of sections by planes making with the axis an angle β is a coaxial right cone generated by a line through the vertex, and inclined to the axis at an angle equal to $\tan^{-1}\tan^2\alpha \operatorname{cotan}\beta$; also that the ratio of the axes of such a section is $[\sin(\alpha+\beta)\sin(\alpha-\beta)]^{\frac{1}{2}}\sec\alpha$; and that, if p is the perpendicular distance of the plane of the section from the vertex of the cone, then the distance of the centre from the foot of p is equal to

$$p \sin\beta\cos\beta \,/\, \sin(\alpha+\beta)\sin(\alpha-\beta).$$

Solution by the Rev. T. C. Simmons, M.A.; Prof. Nash, M.A.; *and others.*



1. Let C be the centre of the section by a plane through AA′; then we have

$$\frac{2\,CQ}{OQ} = \frac{A'Q}{OQ} - \frac{AQ}{OQ},$$

$$\frac{2\sin\theta}{\sin(\beta-\theta)} = \frac{\sin\alpha}{\sin(\beta-\alpha)} - \frac{\sin\alpha}{\sin(\beta+\alpha)}$$

$$= \frac{2\sin^2\alpha\cos\beta}{\sin(\beta-\alpha)\sin(\beta+\alpha)},$$

$$\frac{\sin(\beta-\theta)}{\sin\theta} = \frac{\sin^2\beta - \sin^2\alpha}{\sin^2\alpha\cos\beta};$$

from which we obtain

$$\beta\cot\theta = \frac{\sin^2\beta\cos^2\alpha}{\cos\beta\sin^2\alpha} \text{ or } \tan\theta = \tan^2\alpha\cot\beta.$$

2. Draw MCN perpendicular to axis; then ratio of squares of semi-axes

$$= \frac{MC}{A'C}\cdot\frac{CN}{AC} = \frac{\sin(\beta-\alpha)}{\cos\alpha}\cdot\frac{\sin(\beta+\alpha)}{\cos\alpha}.$$

3. Draw OP perpendicular to A′CA; then we have

$$= CP = A'P - \frac{AA'}{2} = p \cot(\beta - \alpha) - \frac{OA \sin 2\alpha}{2 \sin(\beta - \alpha)}$$

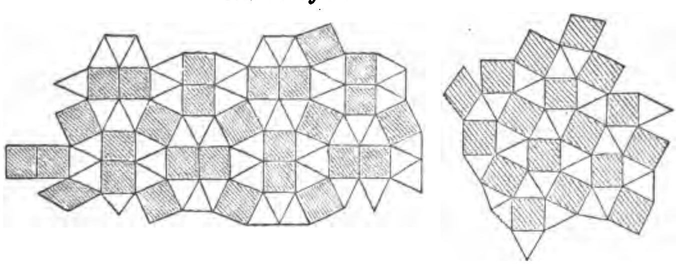
$$= p \cot(\beta - \alpha) - \frac{p}{2} \cdot \frac{\sin 2\alpha}{\sin(\beta + \alpha) \sin(\beta - \alpha)}$$

$$= \frac{p}{2} \cdot \frac{2 \cos(\beta - \alpha) \sin(\beta + \alpha) - \sin 2\alpha}{\sin(\beta + \alpha) \sin(\beta - \alpha)} = \frac{p \sin \beta \cos \beta}{\sin(\beta + \alpha) \sin(\beta - \alpha)}.$$

[The above is the correct formula for determining the figure and position of the "Iris seen in water" (*Phil. Mag.*, 1854).]

6904. (By the Rev. W. A. Whitworth, M.A.)—Required patterns to cover an area with black square tiles, and white equilateral triangular tiles, the side of the square, and the side of the triangle being equal, and the pattern regular; (1) using 2 triangles to 1 square, (2) using 7 triangles to 3 squares.

Solution by the Proposer.



The required patterns are as clearly shown in the annexed figures, which need no explanation.

7406. (By Professor Hudson, M.A.)—Two inclined planes, of the same altitude and inclinations α, β, are placed back to back with an interstice between them. Two weights P, Q are placed one on each plane at the bottom, and connected by a string which passes over two small smooth pulleys at the top and under a movable pulley, weight W, which hangs between the two planes, the free portion of the string being parallel. Find the least value of W, in order that both weights may be drawn up; and, if they arrive at the top at the same time, prove that

$$\frac{4(\sin^2\alpha - \sin^2\beta)}{W} = \frac{2 \sin\alpha + \sin\alpha \sin\beta + \sin^2\beta}{P} - \frac{2 \sin\beta + \sin\alpha \sin\beta + \sin^2\alpha}{Q}.$$

I. *Solution by* R. Rawson; G. B. Mathews, B.A.; *and others.*

Let T = the tension of the string, then the moving forces of W, P, Q are $W-2T,\quad T-P\sin\alpha,\quad T-Q\sin\beta.$

Put x, y for the spaces moved through by P, Q respectively in (t) seconds; then $\frac{1}{2}(x+y)$ will evidently be the space moved through by W in the same time. The differential equations of motion are, therefore, as follows:

$$\frac{d^2x}{dt^2}=g\left(\frac{T}{P}-\sin\alpha\right),\quad \frac{d^2y}{dt^2}=g\left(\frac{T}{Q}-\sin\beta\right)\ \ldots\ldots\ldots(1,\ 2),$$

$$\frac{d^2(x+y)}{dt^2}=2g\left(1-\frac{2T}{W}\right)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

From these we obtain
$$T=\frac{2+\sin\alpha+\sin\beta}{\frac{1}{P}+\frac{1}{Q}+\frac{4}{W}}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Integrating (1), (2), (3), we obtain

$$x=\frac{g}{2}\left(\frac{T}{P}-\sin\alpha\right)t^2,\quad y=\frac{g}{2}\left(\frac{T}{Q}-\sin\beta\right)t^2\ldots\ldots\ldots(5,\ 6),$$

$$x+y=g\left(1-\frac{2T}{W}\right)t^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

The constants of integration are zero, since the motions of P, Q, W are zero when $t=0$. If, therefore, P, Q arrive at the top at the same time,

$$\frac{T}{P}\sin\alpha-\sin^2\alpha=\frac{T}{Q}\sin\beta-\sin^2\beta,\ \text{ or }\ T\left(\frac{\sin\alpha}{P}-\frac{\sin\beta}{Q}\right)=\sin^2\alpha-\sin^2\beta.$$

From (4),

$$\left(\frac{\sin\alpha}{P}-\frac{\sin\beta}{Q}\right)(2+\sin\alpha+\sin\beta)=\left(\frac{1}{P}+\frac{1}{Q}+\frac{4}{W}\right)(\sin^2\alpha-\sin^2\beta);$$

therefore

$$\frac{4(\sin^2\alpha-\sin^2\beta)}{W}=\frac{2\sin\alpha+\sin\alpha\sin\beta+\sin^2\beta}{P}-\frac{2\sin\beta+\sin\alpha\sin\beta+\sin^2\alpha}{Q}.$$

If W be strong enough to draw P, Q up the inclined plane, we have

$$W>2T,\ \text{ or }\ W>\frac{2PQ}{P+Q}(\sin\alpha+\sin\beta).$$

II. *Solution by* D. Edwardes.

Since, in any the same interval, the space passed over by W is half the sum of the spaces described by P and Q (the free portions being parallel), therefore at any instant the velocity of W is half the sum of the velocities of P and Q. Hence the acceleration of W is half the sum of the accelerations of P and Q, that is,

$$2\,\frac{W-2T}{W}=\frac{T-P\sin\alpha}{P}+\frac{T-Q\sin\beta}{Q};$$

whence $T=\dfrac{2+\sin\alpha+\sin\beta}{\frac{1}{P}+\frac{1}{Q}+\frac{4}{W}}$, subject to the condition that T is greater

than the greatest of the quantities P sin α and Q sin β, or W greater than the greatest of the values $\frac{4PQ\sin\alpha}{(2+\sin\beta)Q-P\sin\alpha}$, $\frac{4PQ\sin\beta}{(2+\sin\alpha)P-Q\sin\beta}$.

Let t be the whole time of motion. Then $\frac{2s}{f}=\frac{2s'}{f'}=t^2$, or $f\sin\alpha=f'\sin\beta$; and, substituting for f and f' their values, viz.,

$$g\,\frac{T-P\sin\alpha}{P} \text{ and } g\,\frac{T-Q\sin\beta}{Q},$$

we have the required equation.

7417. (By R. RUSSELL, B.A.)—Show that $A_1, A_2 \dots A_{2n}$ can be found such that, if a certain invariant relation holds between $a_1, a_2 \dots a_{2n}$,

$$A_1(x-a_1)^{2n}+A_2(x-a_2)^{2n}+\dots+A_{2n}(x-a_{2n})^{2n} \equiv P(x-a_1)(x-a_2)\dots(x-a_{2n}).$$

Solution by G. B. MATHEWS, B.A.; SARAH MARKS; *and others.*

Since the left-hand side is rational and homogeneous of degree $2n$ in x, it is sufficient to make it vanish when $x = a_1\,a_2 \dots a_{2n}$ respectively: hence

$$A_2(a_1-a_2)^{2n}+A_3(a_2-a_3)^{2n}+\dots=0,$$
$$A_1(a_2-a_1)^{n}+\dots+A_5(a_2-a_5)^{2n}+\dots=0;$$

whence the invariant relation

$$\begin{vmatrix} 0 & (a_1-a_2)^{2n} & \dots & (a_1-a_{2n})^{2n} \\ (a_2-a_1)^{2n} & 0 & \dots & \dots \\ (a_3-a_1)^{2n} & \dots & \dots & \dots \\ 1 & & & \end{vmatrix} = 0,$$ and the A's are proportional to first minors of this determinant.

7407. (By Professor WOLSTENHOLME, M.A.)—Prove that the three conics $x^2+ay=a^2$, $x^2-y^2=ax$, $y^2-xy=a^2$ have three common points

$$\frac{x}{\sin\frac{2}{7}\pi}=\frac{y}{\sin\frac{3}{7}\pi}=\frac{-a}{\sin\frac{1}{7}\pi},\quad \frac{-x}{\sin\frac{1}{7}\pi}=\frac{y}{\sin\frac{2}{7}\pi}=\frac{a}{\sin\frac{3}{7}\pi},\quad \frac{x}{\sin\frac{3}{7}\pi}=\frac{-y}{\sin\frac{1}{7}\pi}=\frac{a}{\sin\frac{2}{7}\pi};$$

the other common points of them, taken two and two, being $x=y=\infty$; $x=0$, $y=a$; $y=0$, $x=a$.

Solution by R. RAWSON; G. B. MATHEWS, B.A.; *and others.*

Put $x=ax_1$ and $y=ay_1$ in each conic; then we have

$$x_1^2+y_1=1,\quad x_1^2-y_1^2=x_1,\quad y_1^2-x_1y_1=1 \quad\dots\dots\dots\dots(1).$$

The elimination of x_1, y_1 respectively from the 1st and 2nd, 1st and 3rd, and 2nd and 3rd conics gives, respectively,

$$(x_1-1)(x_1^3+x_1^2-2x_1-1)=0,\quad y_1(y_1^3+2y_1^2-y_1-1)=0\dots\dots(2,3),$$
$$x_1(x_1^3+x_1^2-2x_1-1)=0,\quad (y_1-1)(y_1^3+2y_1^2-y_1-1)=0\dots\dots(4,5),$$
$$(x_1^3+x_1^2-2x_1-1)=0,\quad (y_1^3+2y_1^2-y_1-1)=0\dots\dots\dots\dots(6,7).$$

From these equations it is readily inferred that the conics have three common points which are determined by the three roots of the cubics (6) and (7); the other common points, taken two and two, being obvious from the same equations.

From (6) and (7) it follows that $y_1 x_1 = 1$. It now remains to find the roots of the cubic (6). In order to find the roots as given by the learned proposer, it will be necessary to deviate slightly from the ordinary method of solving cubic equations by trigonometrical tables.

If $7\theta = \pi$, then $\sin\theta \left\{64\cos^6\theta - 80\cos^4\theta + 24\cos^2\theta - 1\right\} = 0$ ....(8), the roots of which are π, $\frac{1}{7}\pi$, $\frac{2}{7}\pi$, $\frac{3}{7}\pi$, ... $\frac{6}{7}\pi$ (see Hymers' *Trig.*, p. 89). Equation (8) is by the relation $\cos 2\theta = 2\cos^2\theta - 1$ readily transformed into

$$(2\cos 2\theta)^3 + (2\cos 2\theta)^2 - 2(2\cos 2\theta) - 1 = 0 \quad \text{..............(9)}.$$

Compare (9) with (6), the three roots of which are, therefore,

$$2\cos 2\theta, \quad 2\cos 4\theta, \quad 2\cos 6\theta.$$

Hence $\qquad x = 2a\cos\frac{2}{7}\pi = a\,\dfrac{2\sin\frac{2}{7}\pi\cos\frac{2}{7}\pi}{\sin\frac{2}{7}\pi} = a\,\dfrac{\sin\frac{4}{7}\pi}{\sin\frac{2}{7}\pi} = \dfrac{a\sin\frac{3}{7}\pi}{\sin\frac{2}{7}\pi}.$

The corresponding value of y is obtained as follows:—

$$y = a(1-x_1)(1+x_1) = -a\left(\frac{\sin\frac{4}{7}\pi - \sin\frac{2}{7}\pi}{\sin\frac{2}{7}\pi}\right)\left(\frac{\sin\frac{4}{7}\pi + \sin\frac{2}{7}\pi}{\sin\frac{2}{7}\pi}\right)$$

$$= -a\,\frac{2\cos\frac{3}{7}\pi\sin\frac{1}{7}\pi\;2\sin\frac{3}{7}\pi\cos\frac{1}{7}\pi}{\sin^2\frac{2}{7}\pi} = -a\,\frac{\sin\frac{6}{7}\pi}{\sin\frac{2}{7}\pi} = -\frac{a\sin\frac{1}{7}\pi}{\sin\frac{2}{7}\pi}.$$

Again, $\qquad x = 2a\cos\frac{4}{7}\pi = -2a\cos\frac{3}{7}\pi$

$$= -a\,\frac{2\sin\frac{3}{7}\pi\cos\frac{3}{7}\pi}{\sin\frac{3}{7}\pi} = -a\,\frac{\sin\frac{6}{7}\pi}{\sin\frac{3}{7}\pi} = -\frac{a\sin\frac{1}{7}\pi}{\sin\frac{3}{7}\pi}.$$

The corresponding value of y is

$$y = a(1-x_1)(1+x_1) = a\left(\frac{\sin\frac{3}{7}\pi + \sin\frac{1}{7}\pi}{\sin\frac{3}{7}\pi}\right)\left(\frac{\sin\frac{3}{7}\pi - \sin\frac{1}{7}\pi}{\sin\frac{3}{7}\pi}\right)$$

$$= a\,\frac{2\sin\frac{2}{7}\pi\cos\frac{1}{7}\pi\;2\cos\frac{2}{7}\pi\sin\frac{1}{7}\pi}{\sin^2\frac{3}{7}\pi} = a\,\frac{\sin\frac{2}{7}\pi}{\sin\frac{3}{7}\pi}.$$

Again, $\quad x = 2a\cos\frac{6}{7}\pi = -2a\cos\frac{1}{7}\pi = -a\,\dfrac{2\sin\frac{1}{7}\pi\cos\frac{1}{7}\pi}{\sin\frac{1}{7}\pi} = -\dfrac{a\sin\frac{2}{7}\pi}{\sin\frac{1}{7}\pi}.$

The corresponding value of y is

$$y = a(1-x_1)(1+x_1) = -a\left(\frac{\sin\frac{5}{7}\pi + \sin\frac{1}{7}\pi}{\sin\frac{1}{7}\pi}\right)\left(\frac{\sin\frac{5}{7}\pi - \sin\frac{1}{7}\pi}{\sin\frac{1}{7}\pi}\right)$$

$$= -a\,\frac{2\sin\frac{3}{7}\pi\cos\frac{2}{7}\pi\;2\cos\frac{3}{7}\pi\sin\frac{2}{7}\pi}{\sin^2\frac{1}{7}\pi} = -a\,\frac{\sin\frac{6}{7}\pi\sin\frac{4}{7}\pi}{\sin^2\frac{1}{7}\pi} = -a\,\frac{\sin\frac{3}{7}\pi}{\sin\frac{1}{7}\pi},$$

which proves the beautiful property stated in the question.

7368. (By S. Tebay, B.A.)—Prove the following formula for finding the Dominical or Sunday letter for any given year (given in Woolhouse's excellent little manual on the weights and measures of all nations, in *Weale's Series*)—$L = 2(\frac{1}{4}c)_r + 2(\frac{1}{4}y)_r + 4(\frac{1}{7}y)_r + 1$ (rejecting sevens); where c is the number of completed centuries, and y the years

of the current century; the suffix r indicating *remainder* after each division.

Solution by the PROPOSER.

The Julian year contains 365·25 days, and the Gregorian year 365·2425 days; the difference is $·0075 = \frac{3}{400}$ day, which will amount to 3 days in 400 years. Hence in any proposed year it is only necessary to consider the remainder after division by 400.

The year $Y = 100c + y$, divided by 400, leaves remainder $100\left(\frac{c}{4}\right)_r + y$.

Since $365 = 7 \times 52 + 1$, omitting 7's, we have $2\left(\frac{c}{4}\right)_r + \left(\frac{y}{7}\right)_r$ days.

The number of Julian leap-years is $25\left(\frac{c}{4}\right)_r + \frac{1}{4}\left\{y - \left(\frac{y}{4}\right)_r\right\}$; but the centuries represented by $\left(\frac{c}{4}\right)_r$ are not leap-years in the Gregorian calendar. Therefore the number of leap-years is

$$24\left(\frac{c}{4}\right)_r + \frac{1}{4}\left\{y - \left(\frac{y}{4}\right)_r\right\};$$

whence, adding this to the above, and omitting sevens, the remaining days are

$$5\left(\frac{c}{4}\right)_r + \left(\frac{y}{7}\right)_r + \frac{1}{4}\left\{\left(\frac{y}{7}\right)_r - \left(\frac{y}{4}\right)_r\right\}.$$

Since the first day of the year 1 is Sunday, if we deduct 1 from the above expression, and cast out sevens, the remainder is the number of days after the last Sunday in the year Y. This remainder, taken from 7, gives the Dominical number. Hence, subtract 1, and to avoid fractions multiply by 8, and cast out sevens; the remaining days are

$$5\left(\frac{c}{4}\right)_r - 2\left(\frac{y}{4}\right)_r + 3\left(\frac{y}{7}\right)_r - 1,$$

which, being taken from $7 + 7\left(\frac{c}{4}\right)_r + 7\left(\frac{y}{7}\right)_r$,

gives $$L = 2\left(\frac{c}{4}\right)_r + 2\left(\frac{y}{4}\right)_r + 4\left(\frac{y}{7}\right)_r + 1 \text{ (omitting sevens).}$$

7291. (By D. EDWARDES.)—If the radii of the escribed circles of a triangle are the roots of $x^3 - px^2 + qx - t = 0$, prove that the radii of the escribed circles of its orthocentric triangle are the roots of

$$(pq-t)^3x^3 - 2(pq-t)^2(q^2-pt)x^2 + 16qt^2(pq-t)x - 8t^2[4q^3 - (pq+t)^2] = 0.$$

Solution by the PROPOSER; R. KNOWLES, B.A.; *and others.*

We have $4R = \frac{pq-t}{q}$, $\Delta = tq^{-\frac{1}{2}}$, and since $\Delta = R\sigma$,

therefore $$\Sigma\rho_1\rho_2 = \sigma^2 = \frac{16qt^2}{(pq-t)^2}.$$

Again, $r_1+r_2 = 4R\cos^2\frac{1}{2}C$, and therefore $\rho_1+\rho_2 = 2R\sin^2 C$, thus $\Sigma\rho_1 = 2R\,(1+\cos A\cos B\cos C)$. But $2R\cos C = \frac{r_1+r_2}{2q}(q-r_3^2)$, &c., whence, since $(r_1+r_2)(r_2+r_3)(r_3+r_1) = pq-t$, $\Sigma\rho_1 = \frac{2\,(q^2-pt)}{pq-t}$.

Also, since
$$2R = \Sigma\rho_1 - \frac{\rho_1\rho_2\rho_3}{\Sigma\rho_1\rho_2},$$
therefore
$$\rho_1\rho_2\rho_3 = \Sigma\rho_1\rho_2\left[\frac{2\,(q^2-pt)}{pq-t} - \frac{pq-t}{2q}\right] = \frac{8t^2\,[4q^3-(pq+t)^2]}{(pq-t)^3}.$$

5561. (By J. L. McKenzie.)—Three particles P_1, P_2, P_3 are projected from the same point O in the same vertical plane, and at the same instant. The particle P meets three fixed planes R_1, R_2, R_3, which intersect in O, at distances r_1, r_2, r_3 from O, and at times t_1, t_2, t_3 after projection; and similarly for the other two particles. Prove that

$$\begin{vmatrix} r_1 & r_1' & r_1'' \\ r_2 & r_2' & r_2'' \\ r_3 & r_3' & r_3'' \end{vmatrix} = 0, \quad \text{and} \quad \begin{vmatrix} t_1 & t_1' & t_1'' \\ t_2 & t_2' & t_2'' \\ t_3 & t_3' & t_3'' \end{vmatrix} = 0.$$

Solution by D. Edwardes; J. O'Regan; *and others.*

Let α, β, γ be the inclinations to horizon of the three fixed planes. Let u be the velocity of projection of P_1, and θ its elevation. Then

$$\frac{g\cos\alpha\, r_1}{2u^2\cos^2\theta} = \tan\theta-\tan\alpha,\quad \frac{g\cos\beta\, r_2}{2u^2\cos^2\theta} = \tan\theta-\tan\beta\,\ldots\ldots\ldots(1,\,2),$$
$$\frac{g\cos\gamma\, r_3}{2u^2\cos^2\theta} = \tan\theta-\tan\gamma\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Substituting in (3) the values of $\frac{g}{2u^2\cos^2\theta}$ and $\tan\theta$ found from (1) and (2), and simplifying, we have

$$r_1\cos\alpha\,(\tan\beta-\tan\gamma)+r_2\cos\beta\,(\tan\gamma-\tan\alpha)+r_3\cos\gamma\,(\tan\alpha-\tan\beta) = 0.$$

Similar equations can be found in r_1', r_2', r_3' and r_1'', r_2'', r_3'';

and therefore
$$\begin{vmatrix} r_1 & r_2 & r_3 \\ r_1' & r_2' & r_3' \\ r_1'' & r_2'' & r_3'' \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} r_1 & r_1' & r_1'' \\ r_2 & r_2' & r_2'' \\ r_3 & r_3' & r_3'' \end{vmatrix} = 0.$$

Also since $r_1\cos\alpha = t_1 u\cos\theta$, &c.,

therefore $t_1(\tan\beta-\tan\gamma)+t_2(\tan\gamma-\tan\alpha)+t_3(\tan\alpha-\tan\beta) = 0.$

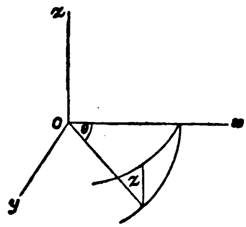
And similar equations in t_1', t_2', t_3' and t_1'', t_2'', t_3'';

wherefore
$$\begin{vmatrix} t_1 & t_2 & t_3 \\ t_1' & t_2' & t_3' \\ t_1'' & t_2'' & t_3'' \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} t_1 & t_1' & t_1'' \\ t_2 & t_2' & t_2'' \\ t_3 & t_3' & t_3'' \end{vmatrix} = 0.$$

6870. (By D. Edwardes.)—A particle under no forces is projected with velocity V along a rough helix; prove that it makes the first n complete revolutions in the time $\dfrac{a}{\mu V \cos^2 \alpha}\left(e^{2\mu n\pi \cos\alpha} - 1\right)$, α being the pitch of the screw, and a radius of cylinder upon which the helix could be drawn.

Solution by G. M. Reeves, M.A.; Sarah Marks; *and others.*

If k be normal pressure and μk the friction, we get (mass of particle being unity)



$$a\,\frac{d^2\theta}{dt^2}\,.\,\tan\alpha = -\mu k \sin\alpha, \text{ and } a\left(\frac{d\theta}{dt}\right)^2 = k;$$

therefore $$\frac{d^2\theta}{dt^2} + \left(\frac{d\theta}{dt}\right)^2 = -\mu\cos\alpha;$$

therefore $$-\frac{1}{\dfrac{d\theta}{dt}} = \mathrm{C} - \mu\cos\alpha\,.\,t,$$

$$\mathrm{C} = -\frac{a}{\mathrm{V}\cos\alpha} \quad \because \frac{d\theta}{dt} = \frac{\mathrm{V}\cos\alpha}{a} \text{ initially};$$

therefore $$\frac{1}{\dfrac{d\theta}{dt}} = \frac{a}{\mathrm{V}\cos\alpha} + \mu\cos\alpha\,.\,t,\quad \frac{dt}{d\theta} = \frac{a + \mu \mathrm{V}\cos^2\alpha\,.\,t}{\mathrm{V}\cos\alpha};$$

therefore $$\frac{1}{\mu\cos\alpha}\,.\log\left(a + \mu\mathrm{V}\cos^2\alpha\,.\,t\right) = \theta + \mathrm{C}.$$

Taking $\theta = 0$, when $t = 0$, $\mathrm{C} = \dfrac{\log a}{\mu\cos\alpha}$,

$$\log\left(a + \mu\mathrm{V}\cos^2\alpha\,.\,t\right) = \mu\cos\alpha\,.\,\theta + \log a,\quad 1 + \frac{\mu\mathrm{V}\cos^2\alpha}{a}\,.\,t = e^{\mu\cos\alpha\,.\,\theta};$$

therefore $$t = \frac{a}{\mu\mathrm{V}\cos^2\alpha}\,.\left(e^{\mu\theta\cos\alpha} - 1\right),$$

and time of describing the angle $2n\pi$ is

$$\frac{a}{\mu\mathrm{V}\cos^2\alpha}\,.\left(e^{2n\pi\cos\alpha} - 1\right).$$

7276. (By S. Tebay, B.A.)—A substance P, suspended from one extremity of a lever, is balanced by a weight Q at the other end, or by a weight Q′ from a second fulcrum: find P, and show that there are two values (P, P′) such that PP′ = QQ′; also, if a be the length of the lever, and a/p the distance between the two fulcrums,

$$\mathrm{Q} = (p-1)^2 m^2 - n^2,\quad \mathrm{Q}' = (p+1)^2 m^2 - n^2,\quad \mathrm{P} = (pm \pm n)^2 - m^2;$$

m, n being any integers prime to one another.

Solution by the PROPOSER.

Let x and $x+\frac{a}{p}$ be the distances of the two fulcrums from P; then

$$x=\frac{Qa}{P+Q'},\quad x+\frac{a}{p}=\frac{Q'a}{P+Q'}.$$

Eliminating x, we have $P^2-[(p-1)Q'-(p+1)Q]P+QQ'=0$. Thus $PP'=QQ'$; and the two values of P are

$$\tfrac{1}{2}[(p-1)Q'-(p+1)Q]\pm\tfrac{1}{2}(Q'-Q)^{\frac{1}{2}}[(p-1)^2Q'-(p+1)^2Q]^{\frac{1}{2}}.$$

Let $$(Q'-Q)[(p-1)^2Q'-(p+1)^2Q]=\frac{n^2}{m^2}(Q'-Q)^2;$$

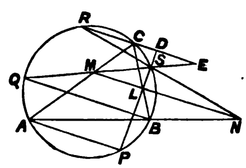
therefore $$[(p-1)^2m^2-n^2]Q'=[(p+1)^2m^2-n^2]Q.$$

Take $Q=(p-1)^2m^2-n^2$, $Q'=(p+1)^2m^2-n^2$; then $P=(pm\pm n)^2-m^2$.

Also $$x=\frac{(p-1)m\mp n}{2pm}a,\quad x+\frac{a}{p}=\frac{(p+1)m\mp n}{2pm}a.$$

6871. (By J. L. McKENZIE, B.A.)—The three sides BC, CA, AB of a triangle are cut by a straight line in L, M, N; and lines drawn through A, B, C, parallel to LMN, cut the circumscribing circle of the triangle ABC in P, Q, R; prove that the lines PL, QM, RN all cut the circle ABC in the same point.

Solution by E. RUTTER; J. O'REGAN; *and others.*



If RN cut the circle at S, the angles CBS, CPS, CAS, CQS are equal; but CRS = SNL, because RC is parallel to LN; hence if LS meet RC at D, the triangles RDS, NLS are obviously similar. Again, suppose PL to meet RC at D′, and RN at S′; then the triangles NLS′, RD′S′ are similar to the triangles RDS, NLS; hence S′ must coincide with S and D′ with D. In like manner, by producing MS to meet RC at E, we prove that QM must pass through S. Hence the theorem is true.

7371. (By W. J. C. SHARP, M.A.)—If ABC be a triangle; O the centre of inscribed circle; D, E, F the points of contact of the sides; and if AO cut EF in A′, BO, FD in B′, and CO, DE in C′; show that the area of the triangle A′B′C′ is $\frac{1}{2}r^2\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$.

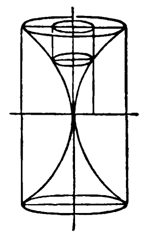
Solution by R. KNOWLES, B.A.; E. J. HENCHIE; *and others.*

The points A′, B′, C′ are evidently the mid-points of EF, FD, DE; therefore $\Delta A'B'C' = \frac{1}{4}\Delta DFE = \frac{1}{4}(ODE + ODF + OFE)$

$$= \tfrac{1}{8}r^2(\sin A + \sin B + \sin C) = \tfrac{1}{2}r^2 \cos\tfrac{1}{2}A \cos\tfrac{1}{2}B \cos\tfrac{1}{2}C.$$

7408. (By the EDITOR.)—If a portion of the parabola $y^2 = 4ax$ cut off by the terminal ordinate c, revolve around the tangent at the vertex, show that the volumes of (1) the solid thus generated, and (2) the greatest cylinder that can be cut therefrom, are $\frac{\pi}{40}\frac{c^5}{a^2}$, $\frac{16\pi}{3125}\frac{c^5}{a^2}$.

Solution by G. B. MATHEWS, M.A.; Prof. MATZ, M.A.; *and others.*



1. Volume $= 2\int_0^c \pi x^2 dy = \frac{\pi}{8a^2}\int_0^c y^4\, dy = \frac{\pi c^5}{40a^2}$.

2. Constructing a cylinder as in figure, its volume

$$= \pi x^2(c-y) = \frac{\pi}{16a^2}y^4(c-y);$$

making this a maximum, we have $y^3(4c-5y) = 0$, whence $y = \frac{4}{5}c$,

and volume $= \frac{\pi}{16a^2}\cdot\frac{4^4}{5^5}\cdot c^5 = \frac{16\pi}{3125}\frac{c^5}{a^2}$.

5682. (By E. W. SYMONS.)—A series of triangles are inscribed in an ellipse so that their orthocentres coincide with the centre of the ellipse; find (1) the locus of their centroids; and (2) prove that the normals at the vertices generally meet in a point.

Solution by D. EDWARDES; J. A. KEALY, M.A.; *and others.*

Let α, β, γ be the excentric angles of the corners of one such triangle; then, since the perpendicular through α to the chord through β, γ passes through the centre, we have $b^2 \sin\alpha = a^2 \tan\frac{1}{2}(\beta+\gamma)\cos\alpha$,

similarly $\quad b^2\sin\beta = a^2\tan\frac{1}{2}(\gamma+\alpha)\cos\beta$ ....(A);

whence, dividing and reducing,

$$\sin\tfrac{1}{2}(\alpha-\beta)\left[\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)\right] = 0;$$

hence, generally, the normals at α, β, γ meet in a point. Again the equation A gives $(a^4+a^2b^2)\cos\alpha\cos\beta+(b^4+a^2b^2)\sin\alpha\sin\beta+a^2b^2=0$.

In the same way, by eliminating β between two such equations,

$$(a^4+a^2b^2)\cos\alpha\cos\gamma+(b^4+a^2b^2)\sin\alpha\sin\gamma+a^2b^2=0.$$

Hence β, γ are the roots of

$$(a^4+a^2b^2)\cos\alpha\cos\theta+(b^4+a^2b^2)\sin\alpha\sin\theta+a^2b^2=0;$$

therefore $$\frac{\cos\frac{1}{2}(\beta+\gamma)}{(a^4+a^2b^2)\cos\alpha}=\frac{\sin\frac{1}{2}(\beta+\gamma)}{(b^4+a^2b^2)\sin\alpha}=\frac{\cos\frac{1}{2}(\beta-\gamma)}{-a^2b^2}\;\ldots\ldots\ldots\ldots(\text{B}).$$

Now if (x, y) are the coordinates of the centroid, by (B),

$$\frac{3x}{a}=\cos\alpha+\cos\beta+\cos\gamma=\cos\alpha-\frac{2a^4b^2\cos\alpha}{(a^2+b^2)(a^4\cos^2\alpha+b^4\sin^2\alpha)};$$

$$\frac{3y}{b}=\sin\alpha+\sin\beta+\sin\gamma=\sin\alpha-\frac{2a^2b^4\sin\alpha}{(a^2+b^2)(a^4\cos^2\alpha+b^4\sin^2\alpha)};$$

hence, squaring and adding, we have, for the locus,

$$\text{I}\left(\frac{x^2}{a^2}=\frac{y^2}{b^2}\right)+\left(\frac{a^2-b^2}{a^2+b^2}\right)^2,$$

an ellipse concentric with and similar to the original.

7231. (By the Editor.)—If A, B, C are candidates for an office, the election to which is in the hands of $8m+1$ electors; and $3m$ votes, together with the casting vote if necessary, are promised to A, and $2m$ votes to B; show that the remaining votes be given so that A may be successful, in $\frac{1}{2}(7m^2+11m+2)$ ways.

Solution by the Rev. J. L. Kitchin, M.A.; E. Rutter; *and others.*

$3m+1$ things can be put into parcels of p, q, r in $(3m+1)!\div(p!\,q!\,r!)$ ways; here subject to the conditions that $p+q+r=3m+1$, and that p, q, r are so chosen that when placed with $3m$, $2m$, 0, for A, B, C respectively, $3m+p$, $2m+q$, r may be such that $3m+p$ may be $>2m+q$ or r; and that, whenever $3m+p$ equals either $2m+q$ or r, the casting vote goes to A; or this case is reckoned also in A's favour. This is the same as $m+p>q$, or $>r-2m$. The total number of ways will equal the sum of the values of $(3m+1)!\div(p!\,q!\,r!)$ with all possible values of p, q, r, taken subject to these conditions, plus the cases in which $m+p=q$ or $3m+p=r$.

The solution is for A, $3m$, $3m+1$, &c.; for B, $2m$, &c.; for C, 0, &c.; and, when counted up, this gives rise to the series

$$m+2+m+3+\ldots+2m+2+2m+1+2m+\ldots+3+2,$$

as is plain on examination; of which the first part

$$=\tfrac{1}{2}(m+1)(3m+2)=\tfrac{1}{2}(3m^2+5m+2),$$

and the second part $=\frac{1}{2}(2m+1)(2+2m)-1=2m^2+3m$; hence the sum is $\frac{1}{2}(7m^2+11m+2)$ ways, as stated.

7405. (By Professor Townsend, F.R.S.) — The rectangular coordinates (x_1, y_1) of a variable point P_1, in a fixed plane, being supposed connected with those (x_2, y_2) of another point P_2, in the same or in another plane, by a relation of the form $f(x_1+iy_1, x_2+iy_2) = 0$, where f is the representative of any function.

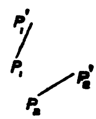
(1) If P_1 describe a curve of small magnitude in its plane, show that P_2 will describe a curve of similar form in its plane.

(2) If P_1 and P_2 be the stereographic projections, of a variable point P on a fixed sphere, upon the planes of the great circles of which any two arbitrary centres of projection O_1 and O_2 on the sphere are the poles: show that (x_1, y_1) and (x_2, y_2) are connected as in (1), and determine the form of f corresponding to the case.

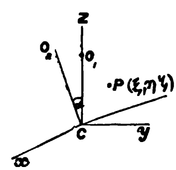
Solution by G. B. Mathews, B.A.; Professor Nash, M.A.; *and others.*

1. If $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, we have

$$\frac{\delta f}{\delta z_1} \cdot dz_1 + \frac{\delta f}{\delta z_2} \cdot dz_2 = 0, \text{ for small variations.}$$



Hence $dz_1 : dz_2 = -\dfrac{\delta f}{\delta z_2} \div \dfrac{\delta f}{\delta z_1}$, a ratio depending only on the values of z_1, z_2; therefore, supposing P_1, P_2 in the same plane, and P_1', P_2' any consecutive positions, the complex (quaternion) ratio $\dfrac{P_1P_1'}{P_2P_2'}$ = constant for small displacements; hence P_1', P_2' describe similar small curves.



2. Take CO_1 for axis of z, O_1CO_2 for plane of yz, and let ξ, η, ζ be the coordinates of P where $\xi^2+\eta^2+\zeta^2 = 1$; then the line O_1P is

$$\frac{x}{\xi} = \frac{y}{\eta} = \frac{z-1}{\zeta-1};$$

hence, putting $z = 0$, we get

$$x_1 = \frac{\xi}{1-\zeta}, \quad y_1 = \frac{\eta}{1-\zeta}.$$

Hence we have $x_1^2+y_1^2 = \dfrac{\xi^2+\eta^2}{(1-\zeta)^2} = \dfrac{1-\zeta^2}{(1-\zeta)^2} = \dfrac{1+\zeta}{1-\zeta}$,

whence $$\zeta = \frac{x_1^2+y_1^2-1}{x_1^2+y_1^2+1}, \quad 1-\zeta = \frac{2}{x_1^2+y_1^2+1},$$

also $$\xi = \frac{2x_1}{x_1^2+y_1^2+1}, \quad \eta = \frac{2y_1}{x_1^2+y_1^2+1}.$$

If $\angle O_1CO_2 = a$, we get

$$x_2 = \frac{\xi}{1-(\zeta\cos a - \eta \sin a)} = \frac{2x_1}{(x_1^2+y_1^2+1)+2y_1 \sin a - (x_1^2+y_1^2-1)\cos a}$$

$$= \frac{x_1}{(x_1^2+y_1^2)\sin^2 \tfrac{1}{2}a + 2y_1 \sin \tfrac{1}{2}a \cos \tfrac{1}{2}a + \cos^2 \tfrac{1}{2}a}$$

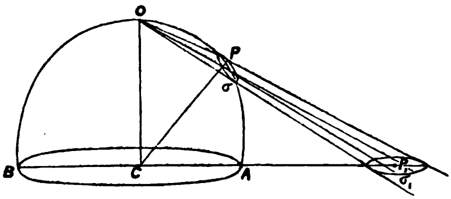
$$= \frac{x_1}{x_1^2 \sin^2 \tfrac{1}{2}a + (y_1 \sin \tfrac{1}{2}a + \cos \tfrac{1}{2}a)^2},$$

$$y_2' = \frac{\zeta \sin \alpha + \eta \cos \alpha}{1 - (\zeta \cos \alpha - \eta \sin \alpha)} = \frac{(x_1^2 + y_1^2 - 1) \sin \alpha + 2y_1 \cos \alpha}{2 x_1^2 \sin^2 \frac{1}{2}\alpha + 2 (y_1 \sin \frac{1}{2}\alpha + \cos \frac{1}{2}\alpha)^2}$$

$$= \frac{(x_1^2 + y_1^2 - 1) \sin \frac{1}{2}\alpha \cos \frac{1}{2}\alpha + y_1 (\cos^2 \frac{1}{2}\alpha \quad \sin^2 \frac{1}{2}\alpha)}{x_1^2 \sin^2 \frac{1}{2}\alpha + (y_1 \sin \frac{1}{2}\alpha + \cos \frac{1}{2}\alpha)^2};$$

$$x_2 + iy_2 = \frac{x_1(\cos^2 \frac{1}{2}\alpha + \sin^2 \frac{1}{2}\alpha) + iy_1(\cos^2 \frac{1}{2}\alpha - \sin^2 \frac{1}{2}\alpha) + i(x_1^2 + y_1^2 - 1) \sin \frac{1}{2}\alpha \cos \frac{1}{2}\alpha}{[(x_1 + iy_1) \sin \frac{1}{2}\alpha + i \cos \frac{1}{2}\alpha]\,[(x_1 - iy_1) \sin \frac{1}{2}\alpha - i \cos \frac{1}{2}\alpha]}$$

$$= \frac{i(x_1 + iy_1) \cos \frac{1}{2}\alpha + \sin \frac{1}{2}\alpha}{(x_1 + iy_1) \sin \frac{1}{2}\alpha + i \cos \frac{1}{2}\alpha} = f(x_1 + iy_1).$$



This may be confirmed by geometry; thus, let a small curve σ be described about P on the sphere, and let σ_1 be its projection on the plane AB.

Then the planes of σ, σ_1 are ultimately perpendicular to CP, CO, and are therefore equally inclined to $OP P_1$, the axis of the slender cone $O(\sigma)$; hence the curves σ, σ_1 are ultimately similar; so for the curves σ, σ_2; therefore σ_1, σ_2 are ultimately similar, so that, if $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ z_2 varies continuously with z_1, and the limit of $\frac{\delta z_2}{\delta z_1}$ is independent of the direction of dz_1; therefore &c.

7440. (By W. J. C. Sharp, M.A.)—Prove that (1) the tangents to the nine-point circle of a triangle, at the points where it meets either side, make angles with that side equal to the difference of the angles adjacent to the side; and (2) the tangent at the middle point makes angles with the other sides which are equal to the opposite angles of the triangle.

Solution by G. Heppel, M.A.; Sarah Marks; *and others.*

1. If AD be the perpendicular from A on BC, then, with D as origin and BC as axis of x, the equation to the nine-point circle is

$$x^2 + y^2 - R \sin(B - C)\, x - R \cos(B - C)\, y = 0;$$

hence the tangent at the origin is $R \sin(B - C)\, x + R \cos(B - C)\, y = 0$.

2. The tangent at the middle point also makes an angle equal to $(B - C)$ with the axis of x, and therefore an angle C with AB.

[If O be the centre of the nine-point circle, and M the middle point of BC, the angle made with BC by the tangent at either D or M is $\sin^{-1} \frac{aD}{2 \text{ radius}} = \sin^{-1} \frac{b \cos C \sim \cos B}{2R} = B \sim C$, which proves (1), and (2) follows as above.]

7355. (By Professor SEITZ, M.A.)—If P, Q, R be three consecutive vertices of a regular polygon of n sides and area Δ, and AB the diameter of the circumscribing circle, and if a triangle be formed by joining three random points on the surface of the polygon: prove that the respective averages of the (1) area and (2) square of area of the triangle are

$$\frac{\Delta}{36n^2}\left\{26\left(\frac{\mathrm{AB}}{\mathrm{PQ}}\right)^2+\left(\frac{\mathrm{AB}}{\mathrm{PR}}\right)^2-9\right\},\quad \frac{\Delta^2}{24n^2}\left\{2\left(\frac{\mathrm{AB}}{\mathrm{PQ}}\right)^2+\left(\frac{\mathrm{AB}}{\mathrm{PR}}\right)^2-1\right\}.$$

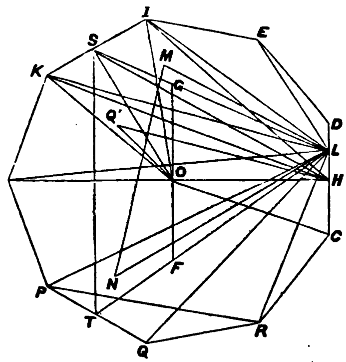
6348 & 6985. (By W. S. B. WOOLHOUSE, F.R.A.S.)—If five points be taken at random on the surface of a regular polygon of n sides, prove (1) that the probabilities that they will be the corners of a (1) convex, (2) regular pentagon, are respectively

$$p_1 = 1-\frac{5}{36n^2}\left\{46\left(\frac{\mathrm{AB}}{\mathrm{PQ}}\right)^2-\left(\frac{\mathrm{AB}}{\mathrm{PR}}\right)^2-15\right\},\quad p_2=\tfrac{7}{12}-\cdot 10\bar{4}\sqrt{5}.$$

Solution by Professor E. B. SEITZ, M.A.

(7355). 1. Let

CDE ... IK ... PQR ...



be the regular polygon, O its centre, and OH its apothegm.

Let Δ_3 be the average area of the triangle when L, one of the points, is taken at random in the perimeter of the polygon, and M, N, the other two points, are taken at random on the surface.

Let OH $=a$, CD $=s$, and $x=$ the apothegm of a regular polygon of n sides, whose centre is O, and whose sides are parallel to those of the given polygon. Then, if L be taken in the perimeter of this polygon, and M, N on its surface, the average area of LMN will be $\Delta_3 x^2 \div a^2$, and we have

$$\Delta_1 = \int_0^a \frac{\Delta_3 x^2}{a^2}\left(\frac{\Delta x^2}{a^2}\right)^2\left(\frac{nsx}{a}\right)dx \div \int_0^a \left(\frac{\Delta x^2}{a^2}\right)^2\left(\frac{nsx}{a}\right)dx = \tfrac{3}{4}\Delta_3.$$

Let the polygon be divided into triangles by drawing lines from L to all the vertices; and let LIK and LPQ be the t^{th} and $(r+t)^{\text{th}}$ triangles respectively; and let S, T be the middle points of IK, PQ; G, F the centres of gravity of LIK, LPQ; HL $=y$, and $\angle$ COH $=\theta=\pi\div n$; then

$$\angle\,\mathrm{HSI}=\angle\,\mathrm{SHD}=t\theta,\quad \mathrm{HS}=2a\sin t\theta,$$

$$\text{area HIK}=\tfrac{1}{2}\mathrm{HS}\,.\,\mathrm{IK}\sin\mathrm{HSI}=as\sin^2 t\theta,$$

$$\text{area LIK}=\mathrm{HIK}\,(\mathrm{HS}-2\mathrm{LH}\cos\mathrm{SHD})\div\mathrm{HS}=s\sin t\theta\,(a\sin t\theta-y\cos t\theta),$$

$$\text{area LPQ}=s\sin(r+t)\,\theta\,[a\sin(r+t)\,\theta-y\cos(r+t)\,\theta],$$

$$\text{area LGF}=\tfrac{4}{9}\mathrm{LST}=\tfrac{4}{9}a\sin r\theta\,[2a\sin t\theta\sin(r+t)\,\theta-y\sin(r+2t)\,\theta].$$

If M, N be taken in the triangle LIK, the average area of LMN will be $\frac{4}{27}$LIK (see WILLIAMSON's *Integral Calculus*, p. 321), and while L ranges

over CD, the sum of all the triangles is

$$S_1 = \int_{-\frac{1}{2}s}^{+\frac{1}{2}s} \tfrac{4}{27}\,(\mathrm{LIK})^3\,dy = \tfrac{4}{27}as^4 \sin^4 t\theta\,(a^2 \sin^2 t\theta + \tfrac{1}{4}s^2 \cos^2 t\theta).$$

Therefore, if M, N both be taken, in succession, in all the triangles of the polygon, the sum of all the triangles LMN is

$$S_2 = \Sigma_{t=1}^{t=n-1} S_1 = \frac{1}{432}\,nas^4\,(20a^2 + s^2) = \frac{2s\Delta^3}{27n^2}\,(4 + \sec^2\theta).$$

If M be taken in the triangle LIK, and N in the triangle LPQ, the average area of LMN will be LGF (see WILLIAMSON's *Cal.*, p. 320), and while L ranges over CD, the sum of all the triangles is

$$S_3 = 2\int_{-\frac{1}{2}s}^{+\frac{1}{2}s} \mathrm{LGF}\,.\,\mathrm{LIK}\,.\,\mathrm{LPQ}\,dy$$

$$= \tfrac{1}{27}a^2s^3 \sin r\theta \sin t\theta \sin (r+t)\,\theta\,[48a^2 \sin^2 t\theta \sin (r+t)\,\theta$$
$$+ 3s^2 \sin 2t\theta \sin 2\,(r+t)\,\theta + 2s^2 \sin^2 r\theta],$$

the integral being doubled to allow for the triangles formed by taking M in LPQ, and N in LIK.

Therefore, for the sum of all the triangles, except those formed by taking M, N in the same triangle of the polygon, we have

$$S_4 = \Sigma_{r=1}^{r=n-2}\,\Sigma_{t=1}^{t=n-r-1}\,S_3 = \tfrac{1}{36}a^4s^3\,\Sigma_{r=1}^{r=n-2}\,[8\,(n-r-1) \sin 2r\theta$$
$$+ (n-r-1) \sin 4r\theta + 12 \operatorname{cosec}\theta \sin r\theta \sin (r+1)\,\theta + 6 \operatorname{cosec}\theta \sin 3r\theta \sin (r+1)\,\theta$$
$$- 6 \operatorname{cosec} 2\theta \sin 2r\theta \sin 2\,(r+1)\,\theta + 2 \operatorname{cosec} 3\theta \sin r\theta \sin 3\,(r+1)\,\theta]$$
$$+ \tfrac{1}{432}a^2s^5\,\Sigma_{r=1}^{r=n-2}\,[4\,(n-r-1) \sin 2r\theta + (n-r-1) \sin 4r\theta$$
$$+ 2 \operatorname{cosec}\theta \sin 3r\theta \sin (r+1)\,\theta + 6 \operatorname{cosec} 2\theta \sin 2r\theta \sin 2\,(r+1)\,\theta$$
$$- 6 \operatorname{cosec} 3\theta \sin r\theta \sin 3\,(r+1)\,\theta]$$
$$= \tfrac{5}{144}na^4s^3\,(7 \cot\theta + \tan\theta) + \tfrac{1}{1728}na^2s^5\,(15 \cot\theta - 7 \tan\theta)$$
$$= \frac{s\Delta^3}{108n^2}\,(105 \operatorname{cosec}^2\theta - 7 \sec^2\theta - 68).$$

Therefore the sum of all the triangles is

$$S = S_2 + S_4 = \frac{s\Delta^3}{108n^2}\,(105 \operatorname{cosec}^2\theta + \sec^2\theta - 36)$$
$$= \frac{s\Delta^3}{27n^2}\,(26 \operatorname{cosec}^2\theta + \operatorname{cosec}^2 2\theta - 9),$$

and, since the whole number of triangles considered is $s\Delta^2$, the average area of the triangle LMN is

$$\Delta_3 = S + s\Delta^2 = \frac{\Delta}{27n^2}\,(26 \operatorname{cosec}^2\theta + \operatorname{cosec}^2 2\theta - 9).$$

$$\therefore \quad \Delta_1 = \tfrac{3}{4}\Delta_3 = \frac{\Delta}{36n^2}\,(26 \operatorname{cosec}^2\theta + \operatorname{cosec}^2 2\theta - 9) = \text{the result stated.}$$

2. By Question 2471 (*Reprint*, Vol. viii., p. 100), we have $\Delta_2^2 = \frac{3}{2}h^2k^2$, where h, k denote the radii of gyration of the polygon round the two principal axes of rotation in its plane. We will find the radius of gyration with respect to OH. Suppose the polygon to be divided into triangles by joining O with all its vertices; and let Q′ be any point in the t^{th} triangle, OIK, $OQ' = z$, $\angle DOH = \beta$, $\angle SOQ' = \phi$, and $z' = a \sec\phi$; then

$$\angle Q'OH = (2t-1)\,\theta + \beta + \phi,$$

and the moment of inertia of OIK is

$$I = \int_{-\theta}^{+\theta}\int_0^{z'} z^2 \sin^2[(2t-1)\theta+\beta+\phi]\, d\phi\, z\, dz$$

$$= \tfrac{1}{4}a^4 \int_{-\theta}^{+\theta} \sin^2[(2t-1)\theta+\beta+\phi] \sec^4\phi\, d\phi$$

$$= \tfrac{1}{2}a^4 \tan\theta \sin^2[(2t-1)\theta+\beta] + \tfrac{1}{6}a^4 \tan^3\theta \cos^2[(2t-1)\theta+\beta].$$

hence $\quad h^2\Delta = \Sigma_{t=1}^{t=n} I = \frac{1}{4}na^4 \tan\theta + \frac{1}{12}na^4 \tan^3\theta = \dfrac{\Delta^2}{12n}(\tan\theta + 3\cot\theta).$

Therefore $\qquad h^2 = k^2 = \dfrac{\Delta}{12n}(\tan\theta + 3\cot\theta),$

and $\quad \Delta_2{}^2 = \frac{3}{2}h^2k^2 = \dfrac{\Delta^2}{96n^2}(9\,\mathrm{cosec}^2\theta + \sec^2\theta - 4)$

$$= \frac{\Delta^2}{24n^2}(2\,\mathrm{cosec}^2\theta + \mathrm{cosec}^2 2\theta - 1) = \text{the result stated.}$$

(6985). By Question 2471, we have

$$p_1 = 1 - \frac{10\Delta_1}{\Delta} + \frac{10\Delta_2{}^2}{\Delta^2} = 1 - \frac{5}{36n^2}(46\,\mathrm{cosec}^2\theta - \mathrm{cosec}^2 2\theta - 15),$$

which is equivalent to the result stated.

(6348.) For the regular pentagon, we have

$$p_2 = \tfrac{7}{12} - \tfrac{47}{450}\sqrt{5} = \tfrac{7}{12} - \cdot 10\dot{4}\sqrt{5}.$$

6044. (By the Editor.)—If the two bottom corners of a leaf of a book, of width c, are turned down in such wise as to meet in a point P, and make one crease twice as long as the other, prove that (1) the equation of the locus of P is $x^2[(c-x)^2+y^2]^3 = 4(c-x)^2(x^2+y^2)^3$, and (2) trace the complete curve thus represented.

Solution by the Rev. T. R. Terry, M.A.; D. Edwardes; *and others.*

1. Let (x, y) be the coordinates of P referred to OA and OB as axes; l, l' the lengths of the creases; θ, ϕ their inclinations to OA; then

$$x = 2l\sin^2\theta\cos\theta, \quad y = 2l\sin\theta\cos^2\theta;$$

hence we have $(x^2+y^2)^{\frac{3}{2}} = 2l\,xy$ ....(1),

$$c - x = 2l'\sin^2\phi\cos\phi, \quad y = 2l'\sin\phi\cos^2\phi;$$

therefore $[(c-x)^2+y^2]^{\frac{3}{2}} = 2l'(c-x)\,y$......(2).

And since, for a general solution, $l' = nl$, equations (1) and (2) give

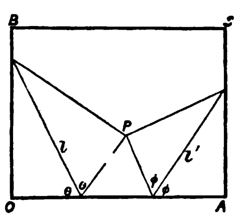


Fig. 1.

$$x^2[(c-x)^2+y^2]^3 = n^2(c-x)^2(x^2+y^2)^3,$$

an equation which, for $n = 2$, gives the stated equation to the locus.

2. At O the curve is approximately $2y^3 = \pm c^2x$, and at A, $y^3 = \pm 2c^2x$;

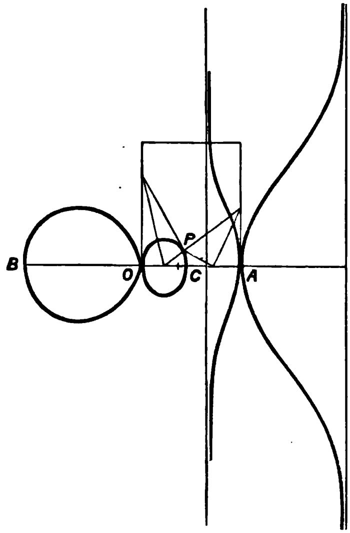


Fig. 2.

thus each branch has a point of inflexion. The asymptotes are $x = \frac{2}{3}c$, $x = 2c$. The sign of $\dfrac{2^{\frac{1}{3}}x^{\frac{2}{3}} - (c-x)^{\frac{2}{3}}}{x^{\frac{2}{3}} - 2^{\frac{2}{3}}(c-x)^{\frac{2}{3}}}$ determines the sign of y^2, and between $x = c(\sqrt{2}-1)$ and $x = \frac{2}{3}$ the curve is imaginary. In the figure, we have $OC = c(\sqrt{2}-1)$, and $OB = c(\sqrt{2}+1)$. The curve is symmetrical with respect to the axis of x.

5426. (By Professor Wolstenholme, M.A.)—Prove that (1) the two points whose distances from A, B, C, the angular points of a triangle, are as sin A, sin B, sin C, and the two whose distances are as cos A, cos B, cos C (one of which is the orthocentre), lie on the straight line joining the centre (O) of the circumscribed circle and the orthocentre (L); (2) the two former points Q, Q′ are real for any acute-angled triangle, and lie in LO produced, their positions being determined by

$$\frac{QL}{OL} = \frac{2k+2}{3k+1}, \quad \frac{Q'L}{OL} = \frac{2-2k}{1-3k},$$

where $k^2 = \dfrac{\cos A \cos B \cos C}{1 + \cos A \cos B \cos C}$; (3) P is always real, and lies in OL

produced, so that OL . OP = square on the radius of the circumscribed circle, and
$$\frac{AP}{AL}=\frac{BP}{BL}=\frac{CP}{CL}=\frac{OP}{R}=\frac{R}{OL}=\frac{1}{(1-8\cos A\cos B\cos C)^{\frac{1}{2}}}.$$
Hence the points will be fixed for all triangles inscribed in the same circle and having the same centroid.

Solution by the PROPOSER.

1. The second point (P) whose distances from A, B, C are as $\cos A : \cos B : \cos C$ (one such point being L, the centre of perpendiculars) lies in GL produced (G being the centroid), so that

$$GP : GL = 1+4\cos A\cos B\cos C : 1-8\cos A\cos B\cos C.$$

If (x, y, z) be areal coordinates of P, we have

$$PA^2 = -a^2yz+b^2z(y+z)+c^2y(y+z) \equiv b^2z^2+c^2y^2+2bc\cos A\, yz,$$
$$\frac{b^2z^2+c^2y^2+2bc\cos A\, yz}{\cos^2 A}=\frac{c^2x^2+a^2z^2+2ca\cos B\, zx}{\cos^2 B}=\frac{a^2y^2+b^2x^2+2ab\cos C\, xy}{\cos^2 C}.$$

But
$$b^2z^2+c^2y^2+2bc\cos A\, yz$$
$$=(x+y+z)(b^2z+c^2y)-(a^2yz+b^2zx+c^2xy)\equiv b^2z+c^2y-S,$$
or
$$\frac{b^2z+c^2y-S}{\cos^2 A}=\frac{c^2x+a^2z-S}{\cos^2 B}=\frac{a^2y+b^2x-S}{\cos^2 C},$$
whence
$$(\cos^2 B-\cos^2 C)(b^2z+c^2y)+(\cos^2 C-\cos^2 A)(c^2x+a^2z)$$
$$+(\cos^2 A-\cos^2 B)(a^2y+b^2x)=0,$$
or the point lies on GL. Let then $x=\lambda+\mu\tan A$, $y=\lambda+\mu\tan B$, $z=\lambda+\mu\tan C$, which for proper values of λ, μ represent any point on GL.

Substituting these in any single equation, the term involving μ^2 disappears, and (taking the first equation) we get
$$\lambda\left[(2b^2+2c^2-a^2)\cos^2 B-(2c^2+2a^2-b^2)\cos^2 A\right]$$
$$+2\mu\left\{[b^2\tan C+c^2\tan B+bc\cos A(\tan B+\tan C)]\cos^2 B\right.$$
$$\left.-[c^2\tan A+a^2\tan C+ca\sin^2 B(\tan C+\tan A)]\cos^2 A\right\}=0,$$
or
$$\lambda(2\sin^2 C-\cos^2 B-\cos^2 A-\cos^2 B+\sin^2 A)$$
$$+2\mu[\sin C\cos C-\tan C(\cos^2 B-\sin^2 A)]=0,$$
or $\lambda[3-2(\cos^2 A+\cos^2 B+\cos^2 C)]+2\mu\tan C(\cos^2 C-\cos^2 B+\sin^2 A)=0$,

or $\lambda(1+4\cos A\cos B\cos C)+2\mu\tan C\,.\,2\sin A\sin B\cos C=0$,

or $\lambda(1+4\cos A\cos B\cos C)+4\mu\sin A\sin B\sin C=0$.

But the point divides GL in the ratio $\mu\tan A\tan B\tan C : 3\lambda$,

or $GP : PL = 1+4\cos A\cos B\cos C : -12\cos A\cos B\cos C$,

whence $GP : GL = 1+4\cos A\cos B\cos C : 1-8\cos A\cos B\cos C$,

or, if O be the centre of the circumscribed circle,
$$OL = OP(1-8\cos A\cos B\cos C).$$

But $OL^2 = R^2(1-8\cos A\cos B\cos C)$, whence $OP\,.\,OL = R^2$, or P, L are reciprocal points with respect to the circumscribed circle (which suggests that there must be a much simpler proof from some other point of view).

Again, if p, q, r be the distances of P from A, B, C,

$$a^2(p^2-q^2)(p^2-r^2)+\ldots-b^2c^2(q^2+r^2)-\ldots+a^2b^2c^2=0,$$

or, if $p=k\cos A$, $q=k\cos B$, $r=k\cos C$,

$$k^4[a^2(\cos^2 A-\cos^2 B)(\cos^2 A-\cos^2 C)+\ldots]-k^2[b^2c^2(\cos^2 B+\cos^2 C)+\ldots]$$
$$+a^2b^2c^2=0,$$

and one value of k^2 is $4R^2$, whence we get for the other

$$4k^2 . R^2=\frac{a^2b^2c^2}{a^2(\sin^2 A-\sin^2 B)(\sin^2 A-\sin^2 C)+\ldots+\ldots},$$

or
$$k^2=\frac{4R^2\sin^2 A\sin^2 B\sin^2 C}{\sin^2 A(\sin^2 A-\sin^2 B)(\sin^2 A-\sin^2 C)+\ldots+\ldots}$$
$$=\frac{4R^2\sin A\sin B\sin C}{\sin A\sin(A-B)\sin(A-C)+\ldots+\ldots},$$

and denominator
$$=\tfrac{1}{2}\sin A[\cos(B-C)-\cos(B+C-2A)+\ldots+\ldots$$
$$=\tfrac{1}{4}(\sin 2B+\sin 2C+\sin 4A-\sin 2A+\ldots]$$
$$=\tfrac{1}{4}[\sin 2A+\sin 2B+\sin 2C+\sin 4A+\sin 4B+\sin 4C]$$
$$=\sin A\sin B\sin C-\sin 2A\sin 2B\sin 2C$$
$$=\sin A\sin B\sin C(1-8\cos A\cos B\cos C),$$

or
$$k^2=\frac{4R^2}{1-8\cos A\cos B\cos C},$$

whence, finally, we obtain

$$\frac{AP}{AL}=\frac{BP}{BL}=\frac{CP}{CL}=\frac{OP}{R}=\frac{R}{OL}=\left(\frac{1}{1-8\cos A\cos B\cos C}\right)^{\frac{1}{2}}.$$

2. The two points Q, Q′, whose distances from A, B, C are as sin A, sin B, sin C, will be similarly determined by the equations

$$\frac{b^2z+c^2y-S}{\sin^2 A}=\frac{c^2x+a^2z-S}{\sin^2 B}=\frac{a^2y+b^2x-S}{\sin^2 C},$$

whence they both satisfy the equation

$$(b^2z+c^2y)(\sin^2 B-\sin^2 C)+\ldots+\ldots=0,$$

the same as before, or both points lie upon GL. Taking $x=\lambda+\mu\tan A$, &c., as before, we get the quadratic for $\lambda:\mu$,

$$\lambda^2[(2b^2+2c^2-a^2)\sin^2 B-(2a^2+2c^2-b^2)\sin^2 A]$$
$$+2\lambda\mu[b^2\tan C+c^2\tan B+bc\cos A(\tan B+\tan C)\sin^2 B$$
$$-c^2\tan A-a^2\tan C-ca\cos B(\tan C+\tan A)\sin^2 A]$$
$$+\mu^2[(b^2\tan^2 C+c^2\tan^2 B+2bc\tan B\tan C\cos A)\sin^2 B$$
$$-(c^2\tan^2 A+a^2\tan^2 C+2ca\tan C\tan A\cos B)\sin^2 A]=0,$$

which, when divided by b^2-a^2, gives

$$2\lambda^2(\sin^2 A+\sin^2 B+\sin^2 C)+2\lambda\mu[2\tan A\tan B\tan C+2\sin A\sin B\sin C)$$
$$+\mu^2\tan^2 A\tan^2 B\tan^2 C=0,$$

or
$$4\lambda^2(1+\cos A\cos B\cos C)+4\lambda\mu\tan A\tan B\tan C(1+\cos A\cos B\cos C)$$
$$+\mu^2\tan^2 A\tan^2 B\tan^2 C=0,$$

or
$$\frac{2\lambda}{\mu\tan A\tan B\tan C}=-1\pm\left(1-\frac{1}{1+\cos A\cos B\cos C}\right)^{\frac{1}{2}}\equiv-1\pm k,$$

whence $\quad GQ:QL=2:3(-1+k),\quad GQ':Q'L=2:-3(1+k).$

Hence the two points Q, Q′ are real when cos A cos B cos C is positive, or for any acute-angled triangle. Measuring distances from O instead of G,

we get $\dfrac{OQ-\frac{1}{3}OL}{OL-OQ} = \dfrac{-2}{3(1-k)}, \quad \dfrac{3OQ-OL}{OL-OQ} = \dfrac{-2}{1-k}, \quad \dfrac{2OQ}{2OL} = \dfrac{-2}{1-3k},$

or $OQ : OL = 2 : 3k-1$, and, since k is less than $\frac{1}{3}$, OQ, OL are of opposite signs, or Q lies in LO produced. Similarly $\dfrac{OQ'}{OL} = \dfrac{-2}{1+3k}$, or Q′ also lies in LO produced. Moreover

$$\frac{1}{OQ} + \frac{1}{OQ'} = -\frac{1}{OL} = \frac{1}{LO} = \frac{2}{2LO},$$

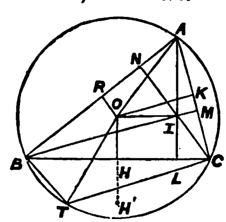
or, if LO be produced to a point R so that OR is twice LO, Q, Q′ will divide OR harmonically. Also $LQ \,.\, LQ' = 9R^2$.

If the equation for the distances from A, B, C be formed as before, the coefficient of k^4 will be the same as before (since $\sin^2 B - \sin^2 C = \cos^2 C - \cos^2 B$, &c.),

whence $\dfrac{AQ \,.\, AQ'}{\sin^2 A} = \dfrac{AP \,.\, AL}{\cos^2 A} = \dfrac{4R^2}{1-8\cos A \cos B \cos C}$, &c.

7428. (By Professor Sylvester, F.R.S.)—If O is the centre of the circle circumscribed about the triangle ABC, and I the intersection of the three perpendiculars from the angles upon the opposite sides of the triangle; prove (1) that the distance of O from any side is half the distance of I from the opposite angle; and hence (2) that OI is the resultant of the three equal forces OA, OB, OC.

Solution by W. G. Lax, B.A.; Margaret T. Meyer; *and others.*



1. Produce AO to cut the circumscribing circle in T, and join TB, TC. Then, since AT is a diameter, ACT is a right angle; hence TC is parallel to OK, and TC = 2OK.

Also CT is parallel to BI, and, since similarly ABT is a right angle, BT is parallel to CI; hence BTCI is a parallelogram; therefore CT = BI and CT = 2OK, therefore BI = 2OK; similarly CI = 2OR, and AI = 2OH.

2. Produce OH to H′ so that OH′ = 2OH; then the resultant of two equal forces OB, OC is 2OH = OH′; hence the resultant of OB, OC, OA is that of OH′ and OA. Now OH′ is parallel and equal to AI, therefore OH′IA is a parallelogram; therefore OI is the resultant of OH′ and OA, that is, of OA, OB, OC.

[If through A, B, C parallels be drawn to BC, CA, AB, the triangle thus formed is similar to ABC and double its linear dimensions; and I being the centre of the circle inscribed in this triangle, we have (1) AI = 2OH, &c.; again, if H, the mid-point of BC, be joined to A, it

will cut OI in G, in such wise that, $AG = 2DG$, and G thus being the centroid, $OI = 3OG$, and therefore (2) is true.

Since O is the centre of the nine-point circle of the larger triangle, the line joining the centre of this circle to that of the circumscribed circle is the resultant of the forces represented by the three radii of the nine-point circle to the middle points of the sides.

It may be shown that the line joining the centres of the inscribed and nine-point circles passes through the centroid, so that the nine-point centre is in OI and indeed bisects it, for $OG = \frac{1}{3}OI = \frac{2}{3}OO'$.

The trilinear equation to this line is

$$\alpha \sin 2A \sin(B-C) + \beta \sin 2B \sin(C-A) + \gamma \sin 2C \sin(A-B) = 0.]$$

7437. (By J. J. Walker, M.A., F.R.S.) — Prove the following formula of reduction for the parts of any spherical triangle ABC:—

$$(\sec a \sin b \cos A - \sin c)^2 + (\sec a \cos b - \cos c)^2 (1 - \operatorname{cosec}^2 a \sin^2 A) = \tan^2 a \cos^2 B \cos^2 C.$$

[The formula is employed, without proof, on p. 69 of Vol. 37 of *Reprints*.]

Solution by D. Edwardes; A. Martin, M.A.; *and others*.

Substituting for cos A its value in terms of the sides, the first term becomes $\cos^2 c \tan^2 a \cos^2 B$, and the second $\sin^2 c \tan^2 a \cos^2 B \left(1 - \dfrac{\sin^2 A}{\sin^2 a}\right)$,

$$\therefore \text{ left side} = \tan^2 a \cos^2 B \left(1 - \frac{\sin^2 c \sin^2 A}{\sin^2 a}\right) = \tan^2 a \cos^2 B (1 - \sin^2 c)$$

$$= \tan^2 a \cos^2 B \cos^2 C.$$

7426. (By Professor Haughton, F.R.S.)—In a work erroneously attributed to Sir Isaac Newton, it is stated, that if two spheres, each one foot in diameter, and of a like nature to the Earth, were distant by but the fourth part of an inch, they would not, even in spaces void of resistance, come together by the force of their mutual attraction in less than a month's time. Investigate the truth of this statement.

Solution by R. Rawson; Professor Matz, M.A.; *and others*.

Let D, D_1, D_2 be the diameters of Earth ($= 7912{\cdot}41 \times 5280$ feet) and of (Earth$_1$), (Earth$_2$) similar in material to the Earth; O the common centre of the three spheres; Q another point such that $OQ = \alpha$; and A, A_1 the attractive forces of the (Earth) and (Earth$_1$) at the point Q; then (Earnshaw's *Dynamics*, p. 318) if ρ is the density of the (Earth) we have

$$A = \frac{\pi\rho D^3}{6\alpha^2}, \quad A_1 = \frac{\pi\rho D_1^3}{6\alpha^2}, \quad \text{whence } \frac{A_1}{A} = \frac{D_1^3}{D^3} \quad \text{......(1, 2, 3).}$$

Let g, g_1 be the accelerating forces at Q produced by A, A_1; then, since accelerating forces are proportional to the moving forces,

$$\frac{g_1}{g} = \frac{A_1}{A} = \frac{D_1^3}{D^3} \text{ from (3), therefore } g_1 = \frac{gD_1^3}{D^3} \quad \text{............(4).}$$

If μ, μ_1 be the accelerating forces of A_1, A_2 at a unit distance from O, where A_2 is the attractive force of the (Earth$_2$), we have

$$\mu = g_1 a^2 = \frac{gD_1^3a^2}{D^3}, \quad \mu_1 = \frac{gD_2^3a^2}{D^3} \quad \text{..................(5, 6).}$$

If, however, Q be on the surface of the (Earth), then $2a = D$, and $g = 32\frac{1}{18}$; hence (5) and (6) become

$$\mu = \frac{gD_1^3}{4D}, \quad \mu_1 = \frac{gD_2^3}{4D} \quad \text{........................(7, 8).}$$

Again, if O, P be the initial positions of (Earth$_1$), (Earth$_2$) respectively, and O′, P′ their positions at the end of (t) seconds; and we put $OP = \beta$, $OO' = x$, $OP' = y$, the dynamical equations of motion are

$$\frac{d^2x}{dt^2} = \frac{\mu_1}{(y-x)^2}, \quad \frac{d^2y}{dt^2} = -\frac{\mu}{(y-x)^2}, \text{ whence } \frac{d^2(y-x)}{dt^2} = -\frac{\mu+\mu_1}{(y-x)^2} \text{...(9,10,11).}$$

This equation, as integrated by Earnshaw, pp. 73, 74, is as follows:—

$$t = \left(\frac{\beta}{2(\mu+\mu_1)}\right)^{\frac{1}{2}} \left\{ \tfrac{1}{2}\beta\pi + [(y-x)(\beta-y+x)]^{\frac{1}{2}} - \tfrac{1}{2}\beta \operatorname{vers}^{-1} \frac{2(y-x)}{\beta} \right\} \text{...(12).}$$

If we say that the spheres come together when they touch, then $y - x = \frac{1}{2}(D_1 + D_2)$, and if we further take $D_1 = D_2 = 1$, then (12) becomes

$$t = \left(\frac{D\beta}{g}\right)^{\frac{1}{2}} \left\{ \tfrac{1}{2}(\beta\pi) + (\beta-1)^{\frac{1}{2}} - \tfrac{1}{2}\beta \operatorname{vers}^{-1} \frac{2}{\beta} \right\} \text{ .... (13).}$$

When $\beta = \frac{49}{8}$ feet, this formula gives for (t) the value 150·91 seconds. This result is so wide apart from a month that I am afraid of having made a slip.

By integrating (11) the velocity of approach is readily seen to be

$$[2(\mu+\mu_1)]^{\frac{1}{2}} \left\{ \frac{1}{y-x} - \frac{1}{\beta} \right\} = \left\{ \frac{32\frac{1}{18}}{7912{\cdot}41 \times 5280 \times 49} \right\}^{\frac{1}{2}} = \cdot 000125 \text{ nearly,}$$

in the case here considered.

7448. (By D. Edwardes.)—If a rectangular hyperbola pass through the centre of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, touch it at a point P, whose eccentric angle is α, and intersect it in Q, R; prove that tangents to the ellipse at Q, R intersect on the straight line

$$b^3x \cos\alpha + a^3y \sin\alpha + ab(a^2 + b^2) = 0.$$

Solution by R. E. Spread, M.A.; Margaret T. Meyer; *and others.*

Let the tangents at Q, R intersect at T (ξ, η); then the equation to any conic passing through Q, R, and touching the ellipse at the point (α), is $a^2y^2 + b^2x^2 - a^2b^2 + \lambda(a^2y\eta + b^2x\xi - a^2b)^2 (ay\sin\alpha + bx\cos\alpha - ab) = 0$;

and the condition of this going through the centre is

$$-a^2b^2 + \lambda \cdot a^3b^3 = 0, \text{ hence } \lambda = (ab)^{-1}.$$

Substituting for λ, and putting $\infty \cdot x^2 + \infty \cdot y^2 = 0$ for rectangular hyperbola, we get $(ab^3 + b^3 \xi \cos a) + (a^3b + a^3\eta \sin a) = 0$; hence the locus is

$$b^3x \cos a + a^3y \sin a + ab\,(a^2 + b^2) = 0.$$

7450. (By R. Tucker, M.A.)—If a circle passing through the focus of a given conic intersects the conic in points $(\theta_1, \theta_2, \theta_3, \theta_4)$, prove that (1) $\Sigma \cos\theta$ is dependent upon the excentricity only; and (2) if the diameter of the circle be inclined to the axis of the conic at an angle $\sin^{-1} l/d$, where $2l$ is the latus rectum and d the diameter of the circle, then one of the angles (θ) is a right angle.

Solution by J. Hammond, M.A.; Margaret T. Meyer; *and others.*

The intersections of the conic and circle are given by the equations $r = \dfrac{l}{1 - e\cos\theta} = d\cos(\theta - a)$; hence, writing x for $\cos\theta$ and m for $\dfrac{l}{d}$,

we have $$\frac{m}{1-ex} = x\cos a + \sin a\,(1-x^2)^{\frac{1}{2}};$$

whence $$\frac{m^2}{(1-ex)^2} - \frac{2m\,x\cos a}{1-ex} + x^2 = \sin^2 a,$$

or $$e^2x^4 - 2ex^3 + (1 - e^2\sin^2 a + 2m\,e\cos a)\,x^2 \\ + 2x\,(e\sin^2 a - m\cos a) + m^2 - \sin^2 a = 0.$$

Therefore $\Sigma\cos\theta$ (or Σx) $= \dfrac{2}{e}$, and if $\sin a = m = \dfrac{l}{d}$, one value of x is zero, or one of the angles is a right-angle [which is also obvious geometrically].

6820. (By H. G. Dawson.)—If $\alpha, \beta, \gamma, \delta$ be the roots of

$$(abcde)(x1)^4 = 0,$$

prove that the equation whose roots are $(\alpha-\beta)^2$, $(\alpha-\gamma)^2$, &c.,

$$\begin{vmatrix} 3, & -z, & -\left(\frac{1}{4}z^2 - \frac{4Hz}{a^2} + \frac{4I}{a^2}\right) \\ z, & \frac{1}{4}z^2 - \frac{4Hz}{a^2} + \frac{I}{a^2}, & \frac{6J}{a^3} \\ \frac{1}{4}z^2 - \frac{4Hz}{a^2} + \frac{I}{a^2}, & \frac{I}{a^2}z + \frac{6J}{a^3}, & -\frac{2J}{a^3}z \end{vmatrix} = 0,$$

where $H = b^2 - ac$, $I = ae - 4bd + 3c^2$, and $J = ace + 2bcd - eb^2 - ad^2 - c^3$.

Solution by the PROPOSER; Prof. NASH, M.A.; *and others.*

Since $(\alpha-\beta)^2+(\gamma-\delta)^2=\dfrac{16H}{a^2}-4\lambda$ and $(\alpha-\beta)^2(\gamma-\delta^2)=\dfrac{4I}{a^2}+\dfrac{24J}{a^3}$,
where λ is a root of $(az)^3-I(az)+2J$,
the roots of $y^2-y\left(\dfrac{16H}{a^2}-4\lambda\right)+\dfrac{4I}{a^2}+\dfrac{24J}{a^3\lambda}=0$ are $(\alpha-\beta)^2$, $(\gamma-\delta)^2$.
Hence the result of eliminating λ between this equation and

$$(a\lambda)^3-I(a\lambda)+2J=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

must be the equation of differences. Now we may write the above equation

$$\lambda^2 y+\left(\frac{y^2}{4}-\frac{4Hy}{a^2}+\frac{I}{a^2}\right)\lambda+\frac{6J}{a^3}=0.$$

Conbining this with (2), we get

$$3\lambda^2-\lambda y-\left(\frac{y^2}{4}-\frac{4Hy}{a^2}+\frac{4I}{a^2}\right)=0,$$

and $$\lambda^2\left(\frac{y^2}{4}-\frac{4Hy}{a^2}+\frac{I}{a^2}\right)+\lambda\left(\frac{I}{a^2}y+\frac{6J}{a^3}\right)-\frac{2J}{a^3}y=0.$$

Eliminate λ and λ^2, and we get the required result.

6990. (By J. HAMMOND, M.A.)—Referring to Professor CAYLEY'S Question 5244, prove that the 16 nodes lie by sixes on sixteen conics, that six of these conics intersect at each node, and that four conicoids may be found, each of which passes through four of the conics and twelve of the nodes, the tetrahedron of reference being self-conjugate with respect to all four of the conicoids.

Solution by the PROPOSER; SARAH MARKS; *and others.*

The sixteen conics are the conics of contact of the sixteen singular tangents, and the four conicoids are

$$Ax^2+By^2+Cz^2+Dw^2=0,\quad Ay^2+Bx^2+Cw^2+Dz^2=0\ldots\ldots(1,\ 2),$$
$$Az^2+Bw^2+Cx^2+Dy^2=0,\quad Aw^2+Bz^2+Cy^2+Dx^2=0\ldots\ldots(3,\ 4).$$

For, by the solution to Question 5244,

$$A\beta^2+B\alpha^2+C\delta^2+D\gamma^2=0,\quad A\gamma^2+B\delta^2+C\alpha^2+D\beta^2=0,$$
$$A\delta^2+B\gamma^2+C\beta^2+D\alpha^2=0.$$

If now we denote the nodes by $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4, d_1, d_2, d_3, d_4$, the order in Quest. 5244 being preserved, and the conics of contact of the singular tangents by A_1, A_2, &c, then it is evident, from the reduced form of the equation to the surface, that

A_1, A_2, A_3, A_4	and the twelve nodes		b, c, d	lie on	(1),
B_1, B_2, B_3, B_4	,,	,,	c, d, a	,,	(2),
C_1, C_2, C_3, C_4	,,	,,	d, a, b	,,	(3),
D_1, D_2, D_3, D_4	,,	,,	a, b, c	,,	(4);

and it may be noticed that the eight intersections of (2), (3), and (4) are the four nodes a, and the four points $(\alpha, -\beta, -\gamma, -\delta)$, $(\alpha, -\beta, \gamma, \delta)$, $(\alpha, \beta, -\gamma, \delta)$, $(\alpha, \beta, \gamma, -\delta)$, which are nodes of the quartic surface, whose equation only differs from that in question by the sign of its last term.

The twelve nodes b, c, d, on (1), are arranged by sixes on the four conics A_1, A_2, A_3, A_4; for the equation to the plane of A_1 being

$$\alpha x + \beta y + \gamma z + \delta w = 0,$$

is satisfied by the coordinates of six nodes, viz.,

$$(\beta, -\alpha, -\delta, \gamma), \quad (\beta, -\alpha, \delta, -\gamma), \quad (\gamma, -\delta, -\alpha, \beta), \quad (\gamma, \delta, -\alpha, -\beta),$$
$$(\delta, -\gamma, \beta, -\alpha), \quad (\delta, \gamma, -\beta, -\alpha);$$

i.e., the six nodes b_2, b_3, c_2, c_4, d_3, d_4 lie on the conic A_1, and the positions of all the nodes may be tabulated as follows:

A_1	$b_2, b_3, c_2, c_4, d_3, d_4$	B_1	$c_3, c_4, d_2, d_4, a_2, a_3$
A_2	$b_1, b_4, c_1, c_3, d_3, d_4$	B_2	$c_3, c_4, d_1, d_3, a_1, a_4$
A_3	$b_1, b_4, c_2, c_4, d_1, d_2$	B_3	$c_1, c_2, d_2, d_4, a_1, a_4$
A_4	$b_2, b_3, c_1, c_3, d_1, d_2$	B_4	$c_1, c_2, d_1, d_3, a_2, a_3$

C_1	$d_2, d_3, a_2, a_4, b_3, b_4$	D_1	$a_3, a_4, b_2, b_4, c_2, c_3$
C_2	$d_1, d_4, a_1, a_3, b_3, b_4$	D_2	$a_3, a_4, b_1, b_3, c_1, c_4$
C_3	$d_1, d_4, a_2, a_4, b_1, b_2$	D_3	$a_1, a_2, b_2, b_4, c_1, c_4$
C_4	$d_2, d_3, a_1, a_3, b_1, b_2$	D_4	$a_1, a_2, b_1, b_3, c_2, c_3$

Also the node a_1 is the intersection of the six conics B_2, B_3, C_2, C_4, D_3, D_4, and so for the others. This is most easily seen from the fact that the node $(\alpha, \beta, \gamma, \delta)$ and the plane $(\alpha, \beta, \gamma, \delta)$ are pole and polar with respect to $x^2 + y^2 + z^2 + w^2 = 0$, so that our tables give by a simple interchange of large and small letters the six singular tangent planes, or, what is the same, the six conics, which intersect at each particular node.

6699. (By Professor TOWNSEND, F.R.S.)—A circular plate of invariable form being supposed, by a small movement of translation in the direction of any diameter, to put in continuous irrotational strain, in the plane of its mass, a surrounding lamina of any incompressible substance extending radially in all directions from its circumference to a fixed boundary at infinity; show that the potential and displacement line-systems of the strain are two systems of circles, passing both through the centre of the plate, and touching respectively its diameters perpendicular and parallel to the direction of its movement.

Solution by the PROPOSER.

Taking for axes of coordinates any pair of rectangular diameters of the plate in its original position, and denoting by r its radius, by l and m the components of its small movement of translation, by ξ and η those of the resulting small displacement of any point xy of the lamina, and by u and v the potential and displacement functions respectively of the strain; then since, by hypothesis, $\xi = 0$ and $\eta = 0$ for all positions of xy at infinity, while for all points on the circumference of the plate in its original

position, that is, for all points satisfying the equation $x^2+y^2=r^2$, they are connected with x and y by the relation $(x+\xi-l)^2+(y+\eta-m)^2=r^2$, or, neglecting small quantities of the second order, by the relation $x\xi+y\eta=lx+my$; we have accordingly, for the solution of the entire problem (including the above particulars) of the strain, first to find, if possible, a potential function u which shall satisfy, at once, the general equation $\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}=0$ throughout the entire extent of the lamina, and the aforesaid particular conditions at its outer and inner boundaries, and then derive from it the corresponding displacement function v in the usual manner. Both functions, in the present case, are found readily as follows. As the function $lx+my$ is homogeneous in x and y, and satisfies for all values of them the equation $\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}=0$, therefore the function $k\,.\,\frac{lx+my}{x^2+y^2}$, where k is any constant, satisfies, for all values of k, the general equation $\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}=0$ throughout the entire extent of the lamina, and the particular conditions $\xi=0$ and $\eta=0$ at its outer boundary, and for the particular value $k=r^2$ that for its inner boundary also. It follows therefore from the aforesaid considerations that $u=r^2\,.\,\frac{lx+my}{x^2+y^2}$, and, by derivation from it in the usual manner, that $v=r^2\,.\,\frac{ly-mx}{x^2+y^2}$; which accordingly are the potential and displacement functions, respectively, of the strain.

Putting $u=c$ and $v=c$, which represent, for different values of c, the potential and displacement line-systems, respectively, of the strain; we get, for the two systems of lines respectively, the equations

$$(x^2+y^2)=\frac{r^2}{c}(lx+my) \text{ and } (x^2+y^2)=\frac{r}{c}(ly-mx),$$

which manifestly establish the two particulars of the question.

To find the principal displacement of the strain at any point xy of the lamina. From the above value of u, by a first differentiation with respect to x and to y, we get at once

$$\left[\left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2\right]^{\frac{1}{2}}=(l^2+m^2)^{\frac{1}{2}}\,.\,\frac{r^2}{x^2+y^2};$$

from which we see that, throughout the entire extent of the strained mass, the principal displacement depends only on the distance from the centre of the plate, and varies from point to point inversely as the square of that distance.

To find the principal dilatation of the strain at any point xy of the lamina. From the same, by a second differentiation with respect to x and to y, we get again at once

$$\left[\left(\frac{d^2u}{dx\,dy}\right)^2-\frac{d^2u}{dx^2}\,.\,\frac{d^2u}{dy^2}\right]^{\frac{1}{2}}=2\,(l^2+m^2)^{\frac{1}{2}}\,.\,\frac{r^3}{(x^2+y^2)^{\frac{3}{2}}};$$

from which we see that, throughout the entire extent of the strained mass, the principal dilatation depends only on the distance from the centre of the plate, and varies from point to point inversely as the cube of that distance.

To find the principal axes of the strain at any point xy of the lamina. From the general equation for the determination of their directions in any case of strain in two dimensions, viz.,

$$(\mu^2-1)\frac{d^2u}{dx\,dy}+\mu\left(\frac{d^2u}{dx^2}-\frac{d^2u}{dy^2}\right)=0,$$

referred in the present case for convenience, to the diameters coinciding with and orthogonal to the movement of the plate, as those of x and y respectively, we have, for the determination of μ at any point xy of the strained mass, the equation $y\,(3x^2-y^2)(\mu^2-1)+x\,(3y^2-x^2)\,2\mu=0$; from which it appears that $\mu=0$ or ∞ when $y=0$ and when $\frac{y}{x}=\pm\sqrt{3}$, and that $\mu=\pm 1$ when $x\equiv 0$ and when $\frac{x}{y}=\pm\sqrt{3}$; and therefore that the principal axes of the strain are parallel and perpendicular to the movement of the plate for all points on its diameter of displacement or on either of the two inclined at angles of 60° to it; and are inclined at angles of 45° to the direction of its movement for all points on its perpendicular diameter, or on either of the two inclined at angles of 60° to it, and therefore at angles of 30° to the direction of its movement.

To find the principal dilatation line-systems of the strain throughout the entire extent of the lamina. Solving for μ from the preceding equation, and substituting for it $\frac{dy}{dx}$ in the result, we get at once, for the differential equations of the two orthogonal systems in question,

$$\frac{d^2u}{dx\,dy}\cdot dy+\frac{d^2u}{dx^2}\cdot dx=+\left[\left(\frac{d^2u}{dx\,dy}\right)^2+\left(\frac{d^2u}{dx^2}\right)\right]^{\frac{1}{2}}\cdot dx,$$

which, by transformation to polar coordinates for which $u=r^{-1}\cos\theta$, become respectively, after the usual reductions,

$$\frac{dr}{r}+\frac{\sin\frac{1}{2}\theta\,.\,d\theta}{\cos\frac{1}{2}\theta}=0,\quad\text{and}\quad\frac{dr}{r}-\frac{\cos\frac{1}{2}\theta\,.\,d\theta}{\sin\frac{1}{2}\theta}=0;$$

the integrals of which, viz., $r^{-\frac{1}{2}}\cos\frac{1}{2}\theta=a^{-\frac{1}{2}}$ and $r^{-\frac{1}{2}}\sin\frac{1}{2}\theta=b^{-\frac{1}{2}}$, where a and b are the two constants of integration, represent, for different values of a and b, two systems of cardioids oppositely situated with respect to each other, having a common cusp at the origin, and a common axis of figure in the displacement diameter of the plate; which accordingly are the line-systems of greatest dilatation and of greatest condensation of the strain.

7380. (By Professor HUDSON, M.A.)—If from the vertex A of a parabola, AY be drawn perpendicular to the tangent at P, and YA produced meet the curve again in Q; prove that PQ cuts the axis in a fixed point.

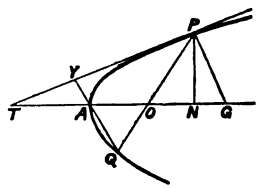
I. *Solution by* A. MARTIN, M.A.; R. KNOWLES, M.A.; *and others.*

The equation to AY is $y=-\frac{y_1}{2a}x$; and this cuts the curve also in the

points $y = -8a^2$, $x = \dfrac{16a^3}{y_1^2}$; hence the equation to PQ is

$$y - y_1 = \frac{y_1 + 8a^2}{x_1 - \dfrac{16a^3}{y_1}}(x - x'), \text{ and when } y = 0 \text{ we have } x = 2a.$$

II. *Solution by the* Proposer.



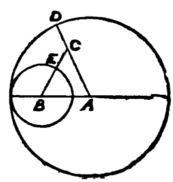
Let the tangent at P and PQ meet the axis in T and O; and draw the normal PG and the ordinate PN; then we have

TA : AO = PO : OQ by property of the parabola,
= OG : AO by similar triangles AQO, GPO.

$\therefore$ OG = AT = AN, $\therefore$ AO = NG = 2AS.

7449. (By C. Bickerdike.)—If a circle A is touched internally by a circle B, and a circle C touches both A and B; show that the locus of the centre of C is an ellipse round the centres of A and B.

Solution by Kate Gale; G. Bayliss, B.A.; *and others.*



Let A, B, C be the centres of the circles; then, since CD = CE, we have

AC + BC = AD + BE = constant;

hence the centre of the third circle describes an ellipse round A, B.

On the Relative Values of the Chessmen. *By* D. Biddle.

The difficulties surrounding this inquiry are numerous and great; and, since many attempts have already been made to arrive at a decisive estimate as to the average value attaching to the several pieces employed in Chess, it might seem reasonable to refer in detail to these previous efforts. But mathematical questions are not to be decided by authority. It may suffice, therefore, to say that some of the ablest papers on the subject have been written by Mr. W. J. C. Miller, and appeared in the *Huddersfield College Magazine*, out of which has grown the *British Chess Magazine*, The pity is, that Mr. Miller had not more scope to carry his investigations to a complete issue.

In estimating the value of the several chessmen, we must duly regard the objects of the game, so far as each player is concerned; viz., to checkmate the adverse king, and to avoid the same on the part of his own king. The capture of adverse pieces is quite of secondary importance, to be engaged in only as tending to accomplish the ultimate object ever to be held in view.

Again, we must bear in mind that the relative value of the several pieces is in great measure a matter of range, the meanest piece being as powerful within its sphere as the highest. Therefore, supposing each piece to be the centre of forces which issue from it in various definite directions, and each square to be the focus of forces which impinge upon or pass through it, these forces must be regarded as equal, except in range.

Except as a passive obstruction, no piece has power over the square on which it stands. A piece standing on a given square obstructs the forces which would otherwise pass through, and receives both these, and also those which terminate at the given square, into itself. The forces thus received by a piece are of two kinds—those coming from friendly, and those from adverse, powers. That which gives intensity to a force coming from an adverse power is the possibility that capture is intended. That which gives virtue to a force coming from a friendly power is the security it affords, that, if such capture take place, reprisals will follow: a piece is thus said to be "guarded." It is of importance, however, to remember that the king can neither be taken nor guarded in this way.

Now, when exchanges take place, it is of moment to know the mean value of the pieces involved. When two pieces stand in a relation of mutual antagonism, the effect of each upon the other is inversely as their respective mean values, other things being equal. In fact, it is a law in Chess, that where the capabilities of two or more pieces are, in respect of any given mode of movement, identical, that which has the least mean value is really the most potent for the time being. Within its sphere, no piece is so powerful as the pawn, simply because no piece can incur such risks.

In estimating the mean value of the several chessmen, we must first consider their range on a clear board, and how this is affected by their position. For there are three ways in which the full powers of the various pieces may be curtailed; viz., (1) by the piece having a position on the board where it has less scope, (2) by the presence of obstructions to its progress in the shape of other pieces, and (3) by its being captured by the enemy and removed from the board altogether.

As to the effect of *position* on the range of a piece, it may be remarked that the castle is the only piece which in any position, on a clear board, commands the same number of squares. Whether in the middle of the board, or at the sides, or even in the corners, the castle, when the board is clear, commands 14 squares, not including the one it occupies. But every other piece varies in range, according to its position, considerably. For the purpose of elucidating this variation in range, it is advantageous to divide the chessboard into a series of concentric square plots. Thus, the whole board consists of 8^2 squares; within that is a set of 6^2 squares; within that, again, a set of 4^2 squares; and finally a central set of 2^2 squares. Having done this, we can assert that only the castles have full power in any part of the 8^2 set; that the king and pawns (as pawns) have their full power only inside the 6^2 set; the knights only inside the 4^2 set; and the queen and bishops only within the very central set of all, the 2^2 set. The bishop of each colour is therefore limited to two squares

when desirous of exhibiting the full extent of his powers; and there are only four squares whence even the queen can take her full command of the board. The following is a tabular statement of the maximum and minimum range on a clear board of each piece:—

	Max.	Min.		Max.	Min.
King	8	3	Bishop	13	7
Queen	27	21	Knight	8	2
Castle	14	14	Pawn	2	1

To find the *average* range of a piece on a clear board, it is necessary to take the sum of its several ranges in every possible position, and thence deduce the mean. We obtain the following results:—

	Average.		Average.
King	$6\frac{9}{16}$	Knight	$5\frac{1}{4}$
Queen	$22\frac{3}{4}$	Pawn, for taking	$1\frac{3}{4}$
Castle	14	Pawn, moving only	$1\frac{1}{8}$
Bishop	$8\frac{3}{4}$		

And, taking the range of the pawn (for capture) as unit, this gives—pawn 1, knight 3, bishop 5, castle 8, queen 13; which is precisely the result arrived at, though in a somewhat different manner, by Mr. Miller.

It must not be supposed, however, that this table gives a strictly true account of the comparative value of the several pieces. There are many ingredients in the character of a piece, and even in the mode of its movement, besides the range, which greatly affect its value. This will appear as we consider the subject of *Obstructions*, which we have classed next to Position, as a main cause of varying powers in any piece.

Under the head of Obstructions is to be classed whatever interferes with the movement of a piece, in any of its characteristic directions, over and beyond mere faults of position which arise from the constitution of the chessboard and the arrangement of the squares. The cause of obstruction is almost invariably some other piece, the only exceptions arising from certain laws of the game, such as that in regard to "castling," in which it is laid down that castling cannot be accomplished if either the king or the particular castle have previously moved, nor if the king be in check, nor if the enemy command either of the two squares involved in the king's movement; viz., that on to, and that over, which he must move. But, although the cause of obstruction is almost invariably some other piece, the piece immediately obstructing is not necessarily an enemy. On the contrary, the most troublesome obstructions are, as a rule, pieces of the same side. This is especially the case at the beginning of the game; and, if such obstructions be traced to their origin, they are found to arise very generally from the peculiarity of the pawn, which, in capturing, takes a different direction from that in which it otherwise moves, and which accordingly cannot itself remove from its onward path that which obstructs it, but is bound to remain where it is so long as the obstacle is before it, unless the enemy place some piece in diagonal juxtaposition with it, or some other piece of its own side can capture the obstacle. It is easy to see how a pawn thus circumstanced will interfere with the free movement of its superiors, and how moves are inevitably lost in making the attempt either to get it out of the way, or to take a circuitous course round it. The pawns which, as a rule, open the game on each side (the kings' pawns) are thus situated from the first. But it is of importance to remember that the advance of the enemy is in this instance retarded no less than our own.

Obstructions may arise when the very square to which we are desirous of moving some piece is (1) already occupied, or (2) commanded by the enemy, or (3) when one or more intervening squares are occupied.

The king, knight, and pawn are never affected by (3); no piece but the king by (2) absolutely,—though other pieces, of course, incur the risk of capture when placed on a square which the enemy commands,—and, except in the case of the pawn, even (1) is an absolute bar only when the piece occupying the square is of the same side as that which it is sought to move thereto.

An adverse piece, even when unguarded, if it come under any but the first category, is an obstruction of a serious nature. For, if it command the desired square, that it is itself unguarded is a matter of very little moment; and if it intervene between the said square and our piece, it necessitates for its capture at least a distinct move, and this may even terminate the game, as where it prevents our at once sheltering our king from check, and where the loss of a move is the loss of everything. But even this is not so galling as to suffer defeat through the stupid obstructiveness of some piece of our own, which, though itself free to move, is in the way of some other piece, and requires time for its removal. Nor is it irrelevant to our subject to remark here how important a factor in the game of Chess is *time*, and what endless complications arise from the fact that each player is rigidly restricted, whenever it becomes his turn to play, to a single distinct act. "Castling" and giving "check by discovery" are no real exceptions to this law.

But it is necessary to observe that obstructions are not only such as prevent the movement of a piece from the square it stands on to some position which is peculiarly valuable. It may so happen that a piece is attacked, either by one of less value or by one like the knight, whose assault it is utterly unable to parry, and where flight is absolutely essential to safety. Then, if there are obstacles in the way to prevent its retreat, its case is hopeless, and the loss may be considerable, unless the enemy's attention can be diverted for a time to some other part of the board, and the play be gradually brought round so as to interfere with those obstacles or the attacking piece. In cases of this kind, the position which the piece occupies on the board is of considerable importance. For instance, the queen, if in the middle of the board, requires eight distinct obstacles to prevent her moving; whereas, at the boundaries, five will suffice, and, in the corners, three. And much the same may be said of the other pieces; but we need speak of no other except the king, who, after all, is chiefly concerned in this matter, since if he be attacked, and there be no retreat open to him, nor any piece to interpose in his defence, he is checkmated, and the game is lost.

The king, if at perfect liberty, has the same number of directions to choose from as the queen, though his move, except in castling, extends to the next square only. But he is differently affected by obstructions; for whereas, in the case of the queen, a distinct obstacle in each direction is requisite to prevent movement absolutely, the same adverse piece will often suffice to prevent movement in several directions in the case of the king, owing to the fact that he moves but one square at a time, and cannot move even to a square that is vacant, if an adverse piece *commands* it.

For the sake of convenience, we may regard the squares surrounding the king, and to any one of which he may move when perfectly free, as forming, together with his own, a set consisting of three rows of three squares each. An adverse piece, whether standing at a distance or near,

may obstruct the king's passage to a variable number of those adjacent squares. Thus, even the pawn, if placed on the square immediately in front of the king, or on that next but one in front, prevents the king from moving to two squares. In the former case, indeed, it appears to obstruct movement in *three* directions; that is, to its own square as well as to two others. But this is an appearance only; for, where the king is concerned, actual obstacles arise only from the presence of pieces of his own side. We have already observed that an adversary may intercept the passage of other pieces, even when itself unguarded, and thus cause the loss of a move; but this cannot happen to the king. Nothing on the part of an adverse piece can obstruct the king's movements but the fact of its *commanding* certain of the squares around him; for he may capture an unguarded adversary that stands on a square adjacent to him, and in the case of one that is guarded the entire obstruction to his appropriating its square lies in the piece or pieces guarding. But it is of importance to remember that to the king an adverse piece is equally obstructive, whether guarded or unguarded, so far as concerns the squares which it commands.

We may next remark that it often becomes necessary to obstruct the movement of the adversary's king before directly "checking" him. If we can hold him to his present position, though he be not in check, or if we can so obstruct his movements as to drive him at any moment wheresoever we will, the most difficult part of the process of checkmating is already accomplished.

We have already given the mean *range* of each piece on a clear board: excluding the king, the same table will serve to indicate the average number of squares from which the adverse king can be "checked" by the several pieces mentioned; since it is obvious, that whatever be the number of squares included in the range of any piece, such also will be the number of distinct positions from which, on a clear board, the same piece could command the said square, or check the king standing on it. It is owing to this fact that Mr. MILLER arrived at the same result as given in the Table of Mean Ranges, he having taken the power of checking as his guide.

But we have shown that to check the king is not always the most serviceable act which a piece can perform. Obstruction of his movements, with or without checking, is equally necessary. Consequently, it is of importance that we discover what power the several pieces have of affecting the adverse king's movements in the various positions in which he may be placed, and what is the average of each piece for the whole board.

Then, since there is an advantage in being able to obstruct as regards several directions by means of one piece,—the queen, for instance, when favourably placed, can check the adverse king, and at the same time prevent his moving in as many as five directions,—it is of importance to calculate the comparative value of each piece in this respect also. But, although it may be doubtful whether a piece which is able to obstruct as regards two directions from three distinct positions, is on the whole of the same value as one that is able to obstruct as regards three directions from two positions, they will in the present paper be held as equivalent.

Then, another fact to be taken into consideration, and which Mr. Miller has partially taken into account under the title of "Safe-checking," is, that some pieces can affect the king from various positions without placing themselves in danger of being captured by him, whilst others

incur this danger more frequently, and require to be guarded. There is, on the contrary, the comparative possibility of interception to be taken into account; for the farther from the king the position whence a piece might affect him, the greater the opening for interposition, except in the case of the knight.

Now, our calculations can be greatly facilitated by the recollection that the chessboard can be divided into similar and equivalent blocks of squares; so that, if we suppose the king to stand in succession upon the several squares of any one such block, and examine the powers of the various pieces in reference to him there, we are in a position to state without further trouble what are their powers in reference to him, when he is placed on the similar squares in the other blocks. Thus, by dividing the board into four equal blocks by two lines drawn through the centre, at right angles to each other, and parallel to the sides, it is not difficult to perceive that the squares of the several blocks are similarly placed as regards the whole board, and consequently that whichever block the king be placed in, if he stand on similar squares, the total power over him of any piece, except the bishops (which can only command squares of the colour they stand on) and the pawns (which have no command at any time over the first two rows of squares), will be the same on a clear board.

The annexed diagram shows which are the similar squares in the several blocks :—

Q	M	H	D	N	O	P	Q
P	L	G	C	I	K	L	M
O	K	F	B	E	F	G	H
N	I	E	A	A	B	C	D
D	C	B	A	A	E	I	N
H	G	F	E	B	F	K	O
M	L	K	I	C	G	L	P
Q	P	O	N	D	H	M	Q

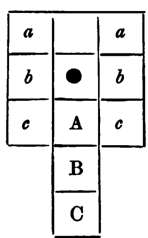
But the effect of similarity in lightening our labours does not end here; for in each block there are squares which, as regards the whole board, are, to all intents and purposes, similarly situated. Thus B = E, C = I, D = N, G = K, H = O, and M = P. The only squares that have none corresponding in the same block are A, F, L, Q. We have therefore only to make our computations in regard to ten squares, to be furnished with data for calculating the powers of the various pieces, so far as their influence upon the movements of the adverse king is concerned, in respect of the whole board. The process is very simple; for, given the power of a piece as regards each of the aforesaid ten squares, we double it in the case of the six that have corresponding squares in the same block; then, adding these several sums to the other four, derived from our examination

regarding the single squares A, F, L, Q, we obtain the sum total for the block; and, dividing this by 16, we obtain the mean power of the piece for the whole board.

We have, however, excepted the bishops and the pawns. The bishops, being each restricted to one colour, meet with strictly similar squares in two only out of the four blocks. The two bishops are equal in power as regards the whole board, though very different as regards any particular square. The plan to adopt, therefore, in regard to them is to examine them together as regards each of the aforesaid ten squares, to add together the twin results in each case, and, treating the several sums as above, to halve the final result. The mean power of each bishop is thus obtained.

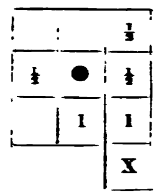
To find the mean power of the pawns, our plan is to take the board by columns. Over two squares in each column the pawns have no command. Moreover, when once they have moved forward, they cannot, as the other pieces can, return to their former position. Our calculations as to the possible effect of the pawns upon the king must necessarily be alike for the four central columns. But the border columns differ considerably, inasmuch as the king when standing on one of their squares can be checked by a pawn on one side only, and the columns next to these are also different, in that the king, when in them, cannot be obstructed by pawns from so many positions. However, the results for the eight columns being added together, the sum must be divided by 64, and again by 8, to give the mean power of the single pawn. In the following calculations the capability of *obstructing* the adverse king, though in regard to one direction only, is held as of equal value with *checking;* and to obstruct in regard to two, three, four, or five directions, as of equal value with the capability of checking from two, three, four, or five distinct positions.

A piece placed in immediate proximity to the king, and therefore in danger, unless supported, of being captured by him, is deemed to be only half as potent as one having equal command from a distance. To take an example, there are three positions on each side of the king (when in the middle of the board), from which the knight can, without checking, obstruct his movements in regard to two directions. The positions of the knight are represented in the accompanying diagram by the squares A, B, C, and the squares commanded are *a, a*; *b, b*; *c, c*, respectively. It is manifest that, in estimating the value of the knight, we must not regard him as equally valuable at all three. At either B or C, he is of double the value that he is at A, because he obstructs to the same extent, and does not require the aid of another piece to prevent his immediate capture.

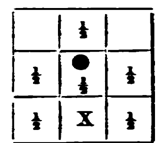


The liability to interposition from which every piece operating from a distance suffers, except the knight, it is somewhat difficult to estimate aright. But there appears to be little doubt that the power of a piece in regard to any square that it is capable of commanding, is, other things being equal, in inverse ratio to the distance that intervenes; for, supposing the object be to check the king, if the chance be *one* that a square adjoining him shall be unoccupied, there will be only *half* the chance that this square and one farther away in the same direction will be vacant, only a *third* of the chance that still another will be vacant, and so on. And the same law holds good with regard to obstructing in all its phases. An illustration will best explain the matter.

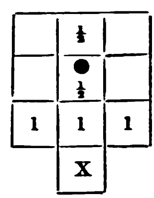
The queen (X), in the position indicated in the diagram, is capable of obstructing the king as regards five directions. Now, we consider the chance to be *one* that she would be able to take up that position, and therefore *one* also that she should command the squares adjacent, since the one fact follows upon the other without fail. But the two adjacent squares indicated may, one or other or both, be occupied, and then, although the king's movements in respect of them are obstructed all the same, the queen's command of the more remote squares is curtailed. The chances in regard to the five squares are as indicated in the diagram, and we consider the queen's power in respect of such a position to be $1 + \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{3} = 3\frac{1}{3}$, instead of 5, as it would have been but for the chance of interposition.



In the following diagram the queen gives check, and obstructs as regards five directions also; but, since she is in close proximity to the king, we regard her power (for reasons given above) as only half what it otherwise would be; that is, 3 instead of 6. This diagram will also serve to show, what must be borne in mind, that the adverse king is never an obstacle in the same sense that other pieces are. The power of the queen to obstruct the king as regards the square beyond him is the same as though he did not stand between.



The strongest position in which the queen, when unsupported, can be placed as regards the adverse king, is shown in the adjoining diagram. Here she insures his being obstructed in regard to three squares in any case, has a half-chance of checking him, and the same of obstructing him as to one square more. Her power, therefore, in respect of such a position, is $1 + 1 + 1 + \frac{1}{2} + \frac{1}{2} = 4$.



In estimating the relative value of the several pieces, so far as their power over the adverse king is concerned, we must suppose them to be equally liable to find the particular squares from which they could affect the king's movements already occupied, and equally liable to be captured by other pieces when placed thereon. Every square on the board must be regarded as neutral in these respects, the chances the same for all.

Let us then proceed to calculate the power of each piece as estimated by the average number of squares from which it can affect the adverse king's movements, and by the average amount of influence it exerts over him from each position. The labour is considerable, but the result is as follows:—

Queen = 74·9848, Knight = 37·2500,
Castle = 41·1729, Bishop = 16·0375, Pawn = $\frac{1}{8}$ (7·3438) = ·917975.

Let it be observed, however, that this table exhibits, at best, the relative value of the several pieces when regarded from one point of view only; that is, as employed offensively against the adverse king. This is their chief office, but it is not the only one, since they have also to defend their own king, and in this aspect their relative value is very different, as we shall presently see.

The exaggerated power which the knight seems to have is due to the fact that he is the only piece that has the full power of checking; that is, ...ing placed in a position to check without being liable to the inter...on of some other piece, and at the same time without incurring the

risk of immediate seizure by the king. The small apparent power of the bishop is due to his being limited to squares of one colour; and the comparative insignificance of the pawn under this head, is due to his inability to move out of his column, except by capturing an enemy, and to his inability to move backwards.

Still, even regarding the pieces simply as weapons of offence, the table ought to be somewhat modified. The power of the several pieces has been deduced in great measure from the number of distinct positions whence each can affect the adverse king. But nothing has yet been said of the time taken by each piece in reaching this limited area from other parts of the board, nor of the varying conditions under which moves are made, so far as freedom is concerned, as the game advances.

The consideration of these two points will lead to a considerable modification of the table.

We will speak of the last mentioned first. There is a gradual diminution in the liability of the pieces to meet with obstructions in their path, which necessarily benefits the queen, castle, and bishop more than the knight and pawn. It is somewhat difficult to determine accurately the rate of diminution, or to deduce therefrom the average chances of obstruction during the whole game. But if we consider the chances to be equal at the beginning of the game, seeing that there are then the same number of squares occupied as unoccupied, they will be as 3 to 1 against a square being occupied when half the pieces are removed, and as 7 to 1 when only eight pieces remain on the board. The mean chance of finding any given square unoccupied, if reckoned from the beginning of the game until only the two kings are left (if such a thing ever occurs), is $5\frac{1}{8}$ to 1, although under the same circumstances the average number of squares unoccupied would be 47, to 17 occupied, or $2\frac{13}{17}$ to 1. But we must not estimate the value of pieces upon any such extreme condition of things as this. The game does not usually last until the board is cleared, nor are pieces removed from the board at regular intervals. Moreover, the chance of finding a square unoccupied increases with no uniform motion, but with rapidly accelerated strides, as piece after piece is removed. We must therefore adjust matters a little. A mean between the degree of freedom enjoyed on a clear board, and that which exists at the beginning of a game, when half the squares are occupied and half unoccupied, seems to be the fairest that can be selected. We have already given the average number of squares commanded by the several pieces in the various positions on the board when clear, and which we call their *range*. And we are now able to give the average number of squares which they command under circumstances of obstruction, when the chances are equal that a square will be occupied or unoccupied.

	Range on clear board.	Scope under equal chance of obstruction.	Mean between Range and Scope.
Queen	22·75	12·15	17·45
Castle	14·00	6·87	10·44
Bishop	8·75	5·12	6·94
Knight	5·25	5·25	5·25
Pawn	1·75	1·75	1·75

In which case, the pawn being 1, the knight is 3, the bishop 4, the castle 6, and the queen 10.

Applying the ratio between the *scope* and the *mean* in the above table to the figures in the table preceding, we find the mean effect produced upon the adverse king by the several pieces to be as follows:—

Queen = 107·7,	Bishop = 21·7,	Pawn = ·918.
Castle = 62·6,	Knight = 37·25,	

As to the extent of the moves of the several pieces, we may observe that that of the knight never varies, but may always be expressed by the length ($=\sqrt{5} = 2{\cdot}236$) of diagonal of a figure two squares in length and one square in breadth. The moves of the other pieces do vary in extent to a very considerable degree; and it is somewhat difficult to apportion the proper value to the different modes of motion. Thus it is a question whether the bishops, who move diagonally, should be allowed the full value of such a movement, as expressed in the distance actually traversed, which, in respect of every square passed over, is $\sqrt{2}$ instead of 1. But we have given the knight the full benefit of a somewhat similar peculiarity, and there is a disadvantage at which the castle, and the queen, in so far as she moves like the castle, stand when compared with the bishop, which must not be disregarded. The chief obstacles to the passage of the superior pieces are necessarily (from their number) the pawns. The pawns move directly forward only, except when capturing. Consequently the castle, half of whose possible moves lie in a line with those of the pawns, is less likely to get rid of the obstruction than is the bishop, who has only the particular square on which the pawn stands obstructed, and who is freed the moment the pawn moves. The bishop can move through pawns when they are not in a row, though they stand on adjacent squares; but the only hope of the castle, under similar circumstances, lies in one of the pawns being bribed to move out of the column in which it stands, by the offer of an adverse pawn, or, if that be not enough, of some superior piece; and, after all, his worst obstacles are pawns of his own colour. It seems only right, therefore, to give the bishop the advantage which accrues from his diagonal motion, and to multiply the squares he moves over by $\sqrt{2}$. Moreover, we may regard the pawn as having, on the average, one opportunity of making the diagonal movement by capture, together with the power of making the two-square move at starting, and as having two single-square moves besides. It will then be fair to calculate the average moves of the bishop and castle by reference to their choice of moves when situated on the several squares of a clear board, and by giving to each possible move an equal probability. That of the queen will be the mean between the two.

The average moves of the various pieces, thus calculated, are as follows:—

Queen = 3·2434,	Knight = 2·2360,	Bishop = 3·4867 = $\sqrt{2}$ (2·4655).
Castle = 3·0000,	Pawn = 1·3540,	

Here the bishop has the pre-eminence, even over the queen, who can move precisely like him, but whose superiority consists in being able also to move like the castle, in which case the squares she passes over do not count for so much, and therefore give her a lower average.

If now we multiply the figures given in the Table of Mean Scope or Command by those now obtained for the average moves, we gain the following estimate—which may be considered a pretty fair one—for the

comparative values of the several pieces for general purposes; that is, of an offensive character :—

Queen = 56·60,	Bishop = 24·20,	Pawn = 2·37.
Castle = 31·32,	Knight = 11·74,	

By raising these figures so as to make the same sum as those given in the Table of Comparative Influence over the adverse king—by multiplying each by 1·8235,—and then striking the difference between the two sets of figures, we obtain the following mean values of the pieces for all offensive purposes :—

Queen = 105·45,	Bishop = 32·92,	Pawn = 2·62;
Castle = 59·86,	Knight = 29·33,	

or, reducing the queen to 10, the values are as follows :—

Queen = 10·00,	Bishop = 3·12,	Pawn = 0·25.
Castle = 5·68,	Knight = 2·78,	

This is the mean comparative power of attack, referred to the queen as standard, rather than to the pawn, whose power is the least capable of being determined with any precision, as we shall see presently. Suffice it now to say, that when we consider the small chance that any particular pawn has of being in a position directly to affect the adverse king's movements, and the very limited number of squares that it can possibly come in contact with, we can readily understand why the average power assigned to the pawn, in its offensive capacity, should be so small.

As to the knight and bishop, it is of importance to remember that at the beginning of the game the knight is considerably superior to the bishop, and that it is not until about five pieces have been removed from the board that the bishop rises superior to him. In the matter of forcing the adverse king to move, the bishop never does rise superior to the knight.

We must now turn to consider the various pieces in their other aspect, viz., that of *defence*.

Defence is of two kinds—by interposition and by guarding. The former is the only method by which the king can be defended, but is rarely of avail unless the interposed piece take up a guarded position; that is, a square next the king, or one commanded by another friendly power. Interposed pieces are liable to be taken. Consequently, when interposition becomes necessary, and we have two or more pieces commanding a square in the enemy's line of attack, we naturally select the piece of least worth, unless there be some very good reason to the contrary. The same law obtains when exchanges occur.

For defensive purposes, therefore, the value of pieces bears a species of inverse ratio to that which they possess as agents of attack. Nevertheless, the ratio is not purely inverse, because the greater the range of a piece, upon which its value in attack mainly depends, the greater also its power of interposing in defence of the king and other pieces. Accordingly, we may regard the value for defensive purposes to be *inversely* as the entire value, and at the same time *directly* as the mean scope under average conditions. The entire value of a piece is the product of its separate values for offensive and defensive purposes, those separate values depending upon qualities that conduce to one end.

Let x be the entire value, a the value as agent of attack, and b the

mean scope; then $\frac{ab}{x} = x$, whence $x = \sqrt{ab}$; consequently, to find the entire value of a piece, we multiply its value as an agent of attack by its mean scope, and take the square-root of the product. This gives the following results:—

Name of Piece.	Value in offensive capacity.	Mean scope under average conditions.	Entire Value.
Queen	10·00	10·0	10·00
Castle	5·68	6·0	5·84
Bishop.........	3·12	4·0	3·53
Knight	2·78	3·0	2·89
Pawn	0·25	1·0	0·50

But we have not taken into account the possibility of the pawn reaching the last row of squares and becoming a queen; and it is the difficulty of determining the value attaching to this possibility that renders the pawn, as we have before intimated, so unfit to be a standard by comparison with which to reckon the values of the other pieces. To *queen* a pawn is a matter of such infrequency, that it seems at first sight scarcely fair to introduce it as a modifying circumstance. On casually opening STAUNTON'S *Chess-Player's Handbook*, out of 24 games examined, only two contain an instance of the kind. In one of these, moreover, the pawn's aggrandisement is followed by instant defeat. And, even supposing that it occurs on the average once in every ten games, this is one pawn out of sixteen becoming queen once in ten games; and, supposing that the promoted pawn enters into the play during a fifth of the game, the mean value of the pawn will only be advanced from 0·5 to 0·512. If it occurred once in every game, and the promoted pawn played as queen during four-fifths of the game, the mean value of the pawn would not be doubled. At the same time, there can be no doubt that, although the probability of the pawn's aggrandisement is small, the possibility alone makes attention to the adverse pawns imperative, and lends a value to these humble agents which they would not otherwise possess. It may not seem unreasonable, therefore, to raise their mean value from ·5 to ·75. But this is quite arbitrary.

We have now accomplished the task we took in hand, and it remains only to express a wish that ere long some means may be devised of accurately registering the progress made during a game of chess. We can score in cricket and tennis and billiards, and positively tell what are good and bad hits; but the progress made in chess, except in so far as the capture of pieces is concerned, is in many instances a matter of the barest conjecture, until the final check puts an end to doubt.

Now, the mean scope of the several pieces, and the average effect they produce on the adverse king's movements, have largely entered as factors into the computation of their mean value. It is reasonable, therefore, to believe that in their actual scope at any time, and the actual command they have over the adverse king's squares, will be found the true solution of the difficulty, care being taken not to neglect the question of defence meantime.

Since writing the above, I have had the advantage of examining, at the Reading Room of the British Museum, several works on the subject of my paper, of which Mr. MILLER has been kind enough to make a brief catalogue for my use. The chief of these are PRATT's or PHILIDOR's *Chess Studies*, and TOMLINSON's *Amusements in Chess*, the latter being little better in this particular than a reprint of the former. In fact, PHILIDOR appears to have been the pioneer in investigations of this sort, and TOMLINSON and STAUNTON have copied his results *verbatim*. The book I saw bore date 1817, and gave the values of the pieces as follows :—

Pawn 1·00, Knight 3·05, Bishop 3.50, Rook 5·48, Queen 9.94.

In the *Westminster Chess Club Papers* for July, 1876, Mr. PEIRCE, a correspondent, gives the following *résumé* of TOMLINSON's treatment of the question (which I find to be a copy of PHILIDOR's) :—(1) Average power to move over open board. (2) Power of preventing pieces occupying any square in a particular line (about *middle* of game). Relative power of commanding squares. (3) Power in choosing what point to select as a position of attack. (4) Power to compel removal of assailed piece. (5) Power in giving mate.

Mr. PEIRCE points out that the various results are *added* to arrive at the final average. I am glad I had not seen the investigations of PHILIDOR or TOMLINSON before making my own independent calculations.

7456. (By Professor WOLSTENHOLME, M.A., D.Sc.)—If $u = 0$ be the rational equation of a quadric referred to rectangular axes, prove that the locus of the point of concurrence of three tangent lines, at right angles to each other two and two, is $\dfrac{d^2 u^{\frac12}}{dx^2} + \dfrac{d^2 u^{\frac12}}{dy^2} + \dfrac{d^2 u^{\frac12}}{dz^2} = 0.$

[The corresponding equation when the coordinate axes are inclined at angles α, β, γ is $\dfrac{d^2 u^{\frac12}}{dx^2} + \dfrac{d^2 u^{\frac12}}{dy^2} + \dfrac{d^2 u^{\frac12}}{dz^2} = 2\cos\alpha \dfrac{d^2 u^{\frac12}}{dy\,dz} + \ldots + \ldots$].

Solution by G. B. MATHEWS, B.A.; T. WOODCOCK, B.A.; *and others.*

$$\delta_x u^{\frac12} = \tfrac12 u^{-\frac12}\delta_x u,\quad \delta_x^2 u^{\frac12} = \tfrac12 u^{-\frac12}\delta_x u - \tfrac14 u^{-\frac32}(\delta_x u)^2 = \frac{1}{4u^{\frac32}}\left[2u\,\delta_x^2 u - (\delta_x u)^2\right],$$

and similarly for $\delta_y^2 u^{\frac12}$, $\delta_z^2 u^{\frac12}$; thus, if $u \equiv (abcdfghlmn \between xyz1)^2$, so that

$$\delta_x^2 u = 2a,\quad \delta_x u = ax + hy + gz + l, \text{ \&c.},$$

the equation $\nabla^2 u^{\frac12}$ becomes in its rational form

$$4(a+b+c)u - (\delta_x u)^2 - (\delta_y u)^2 - (\delta_z u)^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Now the tangent cone from (ξ, η, ζ) to the quadric may be written

$$4(abc\ldots \between \xi\eta\zeta 1)^2 u - [x\delta_\xi u + y\delta_\eta u + z\delta_\zeta u + \ldots]^2 = 0,$$

and, if this have three generators at right angles, the sum of coefficients of

a^2, y^2, z^2 is zero, that is

$$4(a+b+c)(abc\ldots \between \xi\eta\zeta 1)^2-(\delta_\xi u)^2-(\delta_\eta u)^2-(\delta_\zeta u)^2=0,$$

and, considering (ξ, η, ζ) to be current coordinates, this agrees with (1); therefore, &c.

7473. (By R. Rawson.)—If v, u, X are given functions of x, show that $y = y_1 + y_2$ is the complete integral of

$$\frac{d^2y}{dx^2}+\left(v-\frac{du}{u\,dx}\right)\frac{dy}{dx}+\left(\frac{v^2-w^2}{4}+\frac{dv}{2dx}-\frac{v\,dw}{2w\,dx}\right)y=\mathrm{X}\ldots\ldots\ldots(1),$$

where y_1, y_2 satisfies the equations

$$\frac{dy_1}{dx}+\left(\frac{v+w}{2}\right)y_1=-\frac{\mathrm{X}}{w}+\frac{dy_2}{dx}+\left(\frac{v-w}{2}\right)y_2=w\ \ldots\ldots\ldots(2).$$

Solutions by (1) Professor Malet, F.R.S.; (2) *the* Proposer.

1. Consider the more general problem, to find the linear differential equation of which the solution shall be the sum of the solutions of the equations $\frac{dy}{dx}+\mathrm{P}y=\mathrm{Q}$, and $\frac{dy}{dx}+\mathrm{R}y=\mathrm{S}$, where P, Q, R, S are functions of x. We have to eliminate y_1 and y_2 from

$$\frac{dy_1}{dx}+\mathrm{P}y_1=\mathrm{Q},\quad \frac{dy_2}{dx}+\mathrm{R}y_2=\mathrm{S},\quad y=y_1+y_2;$$

or y_1 from the first of these equations, and $\frac{dy}{dx}+\mathrm{R}y+(\mathrm{P}-\mathrm{R})\,y_1=\mathrm{Q}+\mathrm{S}$.

The result is $\frac{d^2y}{dx^2}+\left\{\mathrm{R}+\mathrm{P}-\frac{\mathrm{P}'-\mathrm{R}'}{\mathrm{P}-\mathrm{R}}\right\}\frac{dy}{dx}+\left\{\mathrm{RP}+\frac{\mathrm{R}'\mathrm{P}-\mathrm{P}'\mathrm{R}}{\mathrm{P}-\mathrm{R}}\right\}y$

$$=\mathrm{Q}'+\mathrm{S}'+\mathrm{QR}+\mathrm{PS}-(\mathrm{Q}+\mathrm{S})\frac{\mathrm{P}'-\mathrm{R}'}{\mathrm{P}-\mathrm{R}},\quad \text{where}\quad \mathrm{P}'=\frac{d\mathrm{P}}{dx},\ \&\text{c.}$$

For the values of P, Q, R, S given in question, the sinister of this equation is the same as Proposer's, and if we make the S $=\mathrm{X}/w$, instead of the w in the question, the dexter reduces to X.

2. Differentiating (2), we have

$$\frac{d^2y_1}{dx^2}+\left(\frac{v+w}{2}\right)\frac{dy_1}{dx}+\left(\frac{dv}{2dx}+\frac{dw}{2dx}\right)y_1=-\frac{d\mathrm{X}}{w\,dx}+\frac{\mathrm{X}\,dw}{w^2\,dx},$$

$$\therefore\ \frac{d^2y_1}{dx^2}=\frac{(v+w)\,\mathrm{X}}{2w}-\frac{d\mathrm{X}}{w\,dx}+\frac{\mathrm{X}\,du}{w^2\,dx}+\left\{\frac{(v+w)^2}{4}-\frac{dw}{2dx}-\frac{dv}{2dx}\right\}y_1\ldots(3);$$

$$\frac{d^2y_2}{dx^2}=-\frac{(v-w)\,\mathrm{X}}{2w}+\frac{d\mathrm{X}}{w\,dx}-\frac{\mathrm{X}\,dw}{w^2\,dx}+\left\{\frac{(v-w)^2}{4}+\frac{du}{2dx}-\frac{dv}{2dx}\right\}y_2\ldots(4).$$

But $$\frac{dy}{dx}=\frac{dy_1}{dx}+\frac{dy_2}{dx}=-\frac{(v+w)}{2}y_1-\left(\frac{v-w}{2}\right)y_2\ldots\ldots\ldots\ldots\ldots(5),$$

and
$$\frac{d^2y}{dx^2}=\frac{d^2y_1}{dx^2}+\frac{d^2y_2}{dx^2}$$
$$=\mathrm{X}+\left\{\frac{(v+w)^2}{4}-\frac{du}{2\,dx}-\frac{dv}{2\,dx}\right\}y_1+\left\{\frac{(v-w)^2}{4}+\frac{dw}{2dx}-\frac{dv}{2dx}\right\}y_2\ldots(6).$$

Then
$$\left(v-\frac{dw}{w\,dx}\right)\frac{dy}{dx}$$
$$=\left\{\frac{v\,dw}{2w\,dx}+\frac{du}{2dx}-\frac{v^2}{2}-\frac{uv}{2}\right\}y_1+\left\{\frac{v\,dw}{2w\,dx}-\frac{dw}{2\,dx}-\frac{v^2}{2}+\frac{wv}{2}\right\}y_2\ldots(7),$$
$$\left\{\frac{v^2-w^2}{4}+\frac{dv}{2dx}-\frac{v\,dw}{2w\,dx}\right\}y$$
$$=\left(\frac{v^2-w^2}{4}+\frac{dv}{2dx}-\frac{v\,dw}{2w\,dx}\right)y_1+\left(\frac{v^2-w^2}{4}+\frac{dv}{2dx}-\frac{v\,dw}{2w\,dx}\right)y_2\ldots\ldots(8).$$

Add (6), (7), (8), then the result in the question is obtained. Many interesting cases are included in (1) by giving special values to v, w.

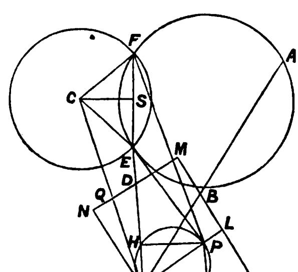
When $v=\dfrac{du}{w\,dx}$, $\quad\dfrac{d^2y}{dx^2}+\left\{\dfrac{d^2u}{2w\,dx^2}-\tfrac{3}{4}\left(\dfrac{dw}{w\,dx}\right)^2-\dfrac{w^2}{4}\right\}y=\mathrm{X}$ is soluble,

when $u=2ax^{2n}$, $\quad\dfrac{d^2y}{dx^2}-\left(\dfrac{n(n+1)}{x^2}+a^2x^{4n}\right)y=\mathrm{X}$ is soluble,

when $u=\dfrac{2a}{\beta}$, $\quad\dfrac{d^2y}{dx^2}+\left\{\tfrac{1}{4}\left(\dfrac{d\beta}{\beta\,dx}\right)^2-\dfrac{d^2\beta}{2\beta\,dx^2}-\dfrac{a^2}{\beta^2}\right\}y=\mathrm{X}$ is soluble.

7462. (By the Editor.)—Through two given points (A, B) draw a circle such that its points of intersection with a given circle (of centre C), and a third given point (P), shall form the vertices of a triangle of given area.

Solution by Rev. T. C. Simmons, M.A.; G. B. Mathews, B.A.; *and others.*



1. Through A, B draw any circle whose radical axis with the circle C meets AB in R; then R must evidently be a point on the chord of intersection of C with the required circle. On RP as diameter draw the circle RHP; take RP.RL = twice the given area; on RL construct a square RLMN; through R draw the line RDEF to meet MN and C so that RD = EF; let RD and RK perpendicular thereto meet the circle RP in H and G; then

FE . PH = RD . RG = RK . RG
 = RP . RL = twice given area,

therefore ΔPEF = given area; hence the required circle is that drawn through the points A, B, E, F.

2. The following is an *algebraic* method for drawing the line RDEF so as to satisfy the condition RD = EF,—a problem which is proposed for a *geometric* solution in Question 7520:—Take R as origin, the line joining R to the centre of C as axis of x, and RL for axis of y; let the equation of C be $x^2+y^2+hxy+2gx+c^2=0$, and assume the equation of RD to be $y=mx$; then, at E and F, $x^2(1+m^2+mh)+2gx+c^2=0$; but the difference between the two values of x here equals a constant $(=2a$, say, $=$ RQ), that is, putting $1+m^2+mh=p$, we have

$$(g^2-c^2p)^{\frac{1}{2}}/p=a \quad \text{or} \quad a^2p^2+c^2p-g^2=0,$$

giving two values of p, from each of which we obtain two values of m.

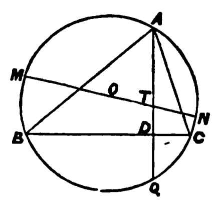
[If we put CE = CF = a = radius of given circle, RC = b, RP = c, $\angle$CRP = α, $\angle$CRE = θ, and k^2 = given area, we shall have

$$k^2=\tfrac{1}{2}\text{EF}\,.\,c\sin(\alpha-\theta)=c\sin(\alpha-\theta)\,(a^2-b^2\sin^2\theta)^{\frac{1}{2}},$$

whence θ and the line REF are determined.]

7457. (By Professor Hudson, M.A.)—If I, O, T are the in-centre, circum-centre, and ortho-centre of a triangle, and r, R the in-radius and circum-radius; prove that $2\,\text{IT}^2-\text{OT}^2=4r^2-\text{R}^2$.

Solution by the Rev. T. C. Simmons, M.A.; R. Nixon, M.A.; *and others.*



Produce OT to meet the circumference in M, N; then we have

$$\text{R}^2-\text{OT}^2=\text{MT}\,.\,\text{TN}=\text{AT}\,.\,\text{TQ}$$
$$=2\text{AT}\,.\,\text{TD}=4r^2-2\,\text{IT}^2$$

(see Quest. 7179, *Reprint*, Vol. xxxix., p. 99); therefore $\quad 2\text{IT}^2-\text{OT}^2=4r^2-\text{R}^2$.

7429. (By Professor Wolstenholme, M.A., D.Sc.)—The rectilinear asymptotes of the curve whose polar equation is $r(\sin\alpha-\sin\theta)=a\sin\alpha\cos\theta$ are $r\sin(\alpha\pm\theta)=a\sin\alpha$. The rectilinear asymptote of the curve

$$r=a\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}\theta) \text{ is } r\cos\theta=2a.$$

Reconcile these results; since, if we put $\alpha=\frac{1}{2}\pi$ in the first equations, we get for the curve the equation $r=a\dfrac{\cos\theta}{1-\sin\theta}=a\tan(\frac{1}{4}\pi+\frac{1}{2}\theta)$, and for the asymptote (the two then coinciding) $r\cos\theta=a$.

Solutions by (1) *the* PROPOSER, (2) G. HEPPEL, M.A., *and others.*

1. In the former curve $r = \infty$ when $\theta = \alpha$ or $\pi - \alpha$, and, taking $u = \frac{1}{r}$, the value of $\frac{du}{d\theta}$, when $\sin\theta = \sin\alpha$ is $-\frac{1}{a\sin\alpha}$; whence the asymptotes are as stated, and the curve is of the form in Fig. 1; α being taken between 0 and $\frac{1}{2}\pi$, O the origin and OA $= a$. The actual value of $\frac{du}{d\theta}$ is

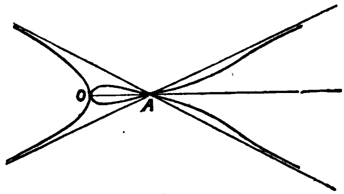


Fig. 1.

$$\frac{1}{a\sin\alpha}\left(-1+\frac{\sin\theta\,(\sin\alpha-\sin\theta)}{\cos^2\theta}\right);$$

and, although the second term generally $= 0$ when $\sin\theta = \sin\alpha$, this is *not* the case when $\alpha = \frac{1}{2}\pi$, its limit then, when $\theta = \alpha$ (or $\pi-\alpha$), being $\frac{1}{2}$, so that $\frac{du}{d\theta}$ is then $-\frac{1}{2a}$, and the asymptote

$$r\cos\theta = 2a.$$

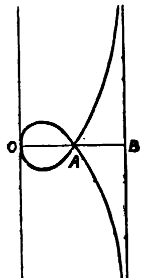


Fig. 2.

The discrepancy thus arises from neglecting to consider this term.

At the same time, it seems singular that, if we suppose α to change gradually from a value $< \frac{1}{2}\pi$ to a value the supplement of the former, the asymptotes should, as α passes through $\frac{1}{2}\pi$, take a sudden leap from A to B (Fig. 2), and back again; which they must do, since the curve is the same for $\pi-\alpha$ as for α.

It should be noticed that, when $\alpha = \frac{1}{2}\pi$, part of the locus of the equation $r(\sin\alpha - \sin\theta) = a\sin\alpha\cos\theta$ is the straight line $\theta = \frac{1}{2}\pi$, drawn through O at right angles to OAB.

2. Assuming the results to have been verified, let a line parallel to the axis of x, and at a distance y from it, have a portion c intercepted between the first curve and its asymptote.

Then $\quad c = x - a - y\cot\alpha = r\cos\theta - a - r\sin\theta\cot\alpha,$

therefore $\quad r(\cos\theta - \sin\theta\cot\alpha) = a + c;$

therefore $$\frac{\sin\alpha-\sin\theta}{\cos\theta-\sin\theta\cot\alpha} = \frac{a\sin\alpha\cos\theta}{a+c};$$

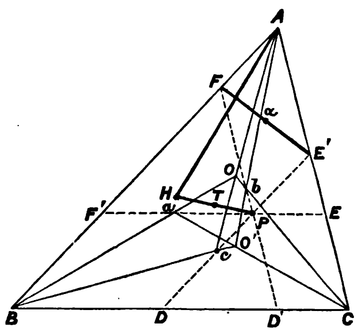
whence we obtain $$c = a\frac{\sin\frac{1}{2}(\alpha-\theta)\sin\theta}{\cos\frac{1}{2}(\alpha+\theta)}.$$

Now, if α have any value less than $\frac{1}{2}\pi$, $\theta = \alpha$ gives $c = 0$, as it should do. But, if $\alpha = \frac{1}{2}\pi$, then $c = a\sin\theta$, and then, if $\theta = \frac{1}{2}\pi$, $c = a$. Hence we are led to conclude that, if two intersecting asymptotes tend to coincide in consequence of the variation of some element, it does not follow that the single asymptote produced by this coincidence will pass through the point of intersection. Here the two, gradually closing like a pair of scissors, seem at the moment of closing to jump together to another position.

7414. (By R. Tucker, M.A.)—If from the "Brocard" points, O, O′, perpendiculars are drawn to the sides of the triangle, and their feet joined, two circumscribed triangles are obtained whose sides respectively make the same angles with the sides of the primitive triangle, and which have a common circumscribed circle; prove that the circumcentre, the centre of the "T. R." circle, and the point P, all lie on a straight line which bisects orthogonally the line OO′ in the centre of the above obtained circle. [The points O, O′ are got by making OBA = OCB = OAC = O′AB = O′BC = O′CA; the point P and the "T. R." circle are defined in the *Educational Times* for June, 1883, p. 178; and the minimum property is established in the *Ladies' and Gentlemen's Diary* for 1859, pp. 52—54.]

Solution by Charlotte A. Scott, B.Sc.

1. Let H be the circumcentre; then, to find the centre of the T. R. circle, we have $\angle$ AFE′ = E′D′F′ (because E′D′F′F is circle) = C by known properties of the points D, E, F, &c.; therefore FE′ is perpendicular to AH. Also, AE′PF being parallelogram, AP bisects FE′ in α. Therefore, if T be the mid-point bisection of PH, Tα bisects FE′ at right angles, therefore the centre of the T. R. circle lies on αT. Similarly it lies on βT and γT, that is, T is centre of the T. R. circle.



2. If from A, B, C parallels be drawn to FD, DE, EF, we have

$\angle$ DFB = DE′F′ = PE′F′,

and $\angle$ D′E′F′ = BAC;

therefore, since E′P, F′P, D′P meet in a point P, the three lines drawn as directed meet in a point O. Also

$\angle$ PE′F′ = PF′D′ = PD′E′,

$\therefore$ $\angle$ OAB = OBC = OCA,

and O is a Brocard-point for ABC; P being the corresponding B-point for E′F′D′. (Call O the *first* B-point.) Similarly lines drawn from ABC parallel to D′E′, &c. meet in a point O′, which is the *other* B-point for ABC, P being the corresponding B-point for FDE. (Call O′ the *second* B-point.)

Let BO and CO′ meet in a, &c. Then $\angle bO'a$ = E′D′F′ = C, and $\angle bOa$ = supplement of FED = supplement of C, therefore baOO′ is a circle, similarly c lies on it.

From H draw a perpendicular to BC, meeting F′E in a'. By (1), bisection PH to bisection DD′ is perpendicular to DD′, therefore PD′ = a'D, *i.e.*, a'D = EC; therefore a'C = ED, *i.e.*, a'B = ED, and is therefore parallel to ED; therefore a' is same as point called a above; and therefore HaP, HbP, HcP are all right angles, *i.e.*, circle abcOO′ is on HP as diameter, therefore the centre of Brocard's circle is at the mid-point of HP.

Now, P is the second B-point for FDE, and it is also the first B-point for E′F′D′. These triangles are equal in all respects, and have the same circumcentre; therefore the distance of the first B-point in each from circumcentre must be same. But the distance of the second B-point in

one from circumcentre = distance of first B-point in other from circumcentre. Therefore in any triangle the two B-points will be at same distance from circumcentre, therefore HO = HO′, therefore OO′ is perpendicular to, and bisected by, HP.

7441. (By R. Russell, B.A.)—If from a point (x_1, y_1) four normals be drawn to $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, prove that (1) the equation of the conic going through x_1, y_1, and the four centres of curvature on the normals, is

$$a^2x^2 + b^2y^2 + \frac{c^4xy}{x_1y_1} - \frac{b^2y_1^2}{x_1}x - \frac{a^2x_1^2}{y_1}y - c^4 = 0;$$

and (2) if $\omega^3 = 1$, the discriminant of this is

$$(a^2x_1^2 + b^2y_1^2 - c^4)\,(a^2x_1^2\omega + b^2y_1^2\omega^2 - c^4)\,(a^2x_1^2\omega^2 + b^2y_1^2\omega - c^4).$$

Solution by E. W. Symons, M.A.; R. W. Hogg, B.A.; *and others.*

The four centres of curvature are the four points of contact of the tangents from (x_1, y_1) to the evolute whose equation is $(ax)^{\frac23} + (by)^{\frac23} = c^{\frac43}$, or rationally $\qquad (a^2x^2 + b^2y^2 - c^4)^3 + 27a^2b^2c^4x^2y^2 = 0$ ....(1).

Also the equation of a tangent to the evolute at (x, y) is

$$\frac{a^{\frac23}\xi}{x^{\frac13}} + \frac{b^{\frac23}\eta}{y^{\frac13}} = c^{\frac43} \quad (\xi,\ \eta \text{ being current coordinates}).$$

If this tangent pass through (x_1, y_1), we have

$$\frac{a^{\frac23}x_1}{a^{\frac13}} + \frac{b^{\frac23}y_1}{y^{\frac13}} = c^{\frac43},$$

or, rationally, $\quad (a^2x_1^3y + b^2y_1^3x - c^4xy)^3 + 27a^2b^2c^4x_1^3y_1^3x^2y^2 = 0$ ....(2).

(1) and (2) are then two curves passing through the four centres of curvature. Multiplying (1) by $x_1^3y_1^3$ and subtracting (2), we get, after extracting the cube root, $x_1y_1(a^2x^2 + b^2y^2 - c^4) = a^2x_1^3y + b^2y_1^3x - c^4xy$; a conic through the centres of curvature, and which is obviously the conic required for its equation is also satisfied by (x_1, y_1). Its discriminant is

$$-a^2b^2c^4x_1^3y_1^3 + \tfrac14 a^2b^2c^4x_1^3y_1^3 - \tfrac14 a^6x_1^7y_1 - \tfrac14 b^6x_1y_1^7 + \tfrac14 c^{12}x_1y_1$$

$$\equiv -\tfrac14 x_1y_1(a^6x_1^6 + b^6y_1^6 - c^{12} + 3a^2b^2c^4x_1^2y_1^2)$$

$$\equiv -\tfrac14 x_1y_1(a^2x_1^2 + b^2y_1^2 - c^4)(a^2x_1^2\omega + b^2y_1^2\omega^2 - c^4)(a^2x_1^2\omega^2 + b^2y_1^2\omega - c^4)$$

if $\omega^3 = 1$.

[If θ be the eccentric angle of one of the feet of normals from (x', y'), we have $\qquad a'x\sin\phi - b'y\cos\phi - c^2\sin\phi\cos\phi = 0,$

$\therefore\ a^3x'^3\sin^3\phi - b^3y'^3\cos^3\phi - c^6\sin^3\phi\cos^3\phi - 3abc^2x'y'\sin^2\phi\cos^2\phi = 0;$

but $\qquad \sin^6\phi + \cos^6\phi = 1 - 3\sin^2\phi\cos^2\phi;$

hence, substituting $x = \frac{c^2}{a}\cos^3\phi,\quad y = -\frac{c^2}{b}\sin^3\phi,$

we have the equation in (1), whose discriminant is that stated in (2).]

4675. (By MORGAN JENKINS, M.A.)—Show that the number of pairs of numbers which have a given number G for their greatest common measure, and another number L (of course, a multiple of G) for their least common multiple, is 2^{n-1}, where n is the number of prime bases the product of whose powers is L/G.

Solution by W. J. CURRAN SHARP, M.A.

If A, B be any two quantities such as are required, we have $A = aG$, $B = bG$, $L = abG$, a and b being prime to each other; and the problem is, in how many ways L/G may be divided into two factors prime to each other. Let $L/G = \alpha^r . \beta^s . \gamma^t \ldots$, where α, β, γ, &c. are primes; then the sets of divisions 1 and $\alpha^r\beta^s\gamma^t\ldots$, α^r and $\beta^s\gamma^t\ldots$, β^s and $\alpha^r\gamma^t$, &c., $\alpha^r\beta^s$ and $\gamma^t\ldots$, &c., which give $1 + n + \dfrac{n(n-1)}{1 . 2} + \ldots = 2^n$ partitions, each of which occurs twice, and therefore the number of ways is 2^{n-1}.

7144. (By Professor TOWNSEND, F.R.S.)—A concylic tetrad of foci of a system of bicircular quartic curves in a plane being supposed given; construct geometrically, for a given point in the plane,

(*a*) The directions of the two curves of the system that pass through it;

(*b*) Their remaining seven points of intersection at finite distances in the plane.

Solution by the PROPOSER.

If P, Q, R, S be the four foci of the given tetrad, O the centre of their containing circle C_0, X, Y, Z the three points of intersection of the three pairs of lines QR and PS, RP and QS, PQ and RS, and C_X, C_Y, C_Z the three circles orthogonal to C_0 having their centres at X, Y, Z; then, for an arbitrary point A in the plane, by known properties of confocal quartics of the bicircular class, if U and U′, V and V′, W and W′ be its six circles of connexion with QR and PS, RP and QS, PQ and RS, and if B, C, D, E be its four inverses with respect to the four circles C_0, C_X, C_Y, C_Z, and F, G, H the three inverses of any one of them B with respect to the remaining three circles C_X, C_Y, C_Z, the two curves of the system passing through A bisect internally and externally the three angles between the three pairs of connecting circles U and U′, V and V′, W and W′, and intersect again at the seven points B, C, D, E, F, G, H in the plane.

7216. (By F. MORLEY, B.A.)—If tangents to two similar epicycloids include a constant angle, prove that a straight line through their intersection, making a constant angle with either, will envelope a similar epicycloid.

Solution by the PROPOSER.

Let a, b, c be the sides of the evanescent triangle formed by the three straight lines; ρ_1, ρ_2, ρ_3 the radii of curvature of their envelopes, then

$$a\rho_1 + b\rho_2 + c\rho_3 = 0 \quad\text{.................................(1).}$$

The equation to an epicycloid is of the form $\rho = \text{A} \sin(a\omega + \beta)$. Let the two be $\quad \rho_1 = \text{A}_1 \sin(a\omega + \beta_1), \quad \rho_2 = \text{A}_2 \sin(a\omega + \beta_2)$.

Substituting in (1), we find for the envelope of the third side

$$\rho_3 = -\text{B}_1 \sin(a\omega + \beta_1) - \text{B}_2 \sin(a\omega + \beta_2) = \text{A}_3 \sin(a\omega + \beta_3),$$

if $\text{B}_1 \cos\beta_1 + \text{B}_2 \cos\beta_2 = -\text{A}_3 \cos\beta_3$, $\text{B}_1 \sin\beta_1 + \text{B}_2 \sin\beta_2 = -\text{A}_3 \sin\beta_3$.

This proves the property.

Consider a special case. Take two consecutive tangents to the *same* epicycloid; the envelope of the line bisecting the angle between them is the evolute of the curve. Hence the evolute of a cycloidal curve is a similar curve.

[Mr. WILLIAMSON has shown (*Reprint*, Vol. xxxiii., p. 67, and Vol. xxxvi., p. 63) that if two sides of a triangle moving in a plane envelop involutes of a circle, the third will also. This, of course, holds for any involute: the n^{th} involute is defined by

$$\frac{d^n\rho}{d\omega^n} = \text{constant} = k;$$

and from $a\rho_1 + b\rho_2 + c\rho_3 = 2\Delta$ we get

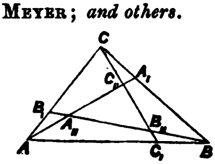
$$a\frac{d^n\rho_1}{d\omega^n} + b\frac{d^n\rho_2}{d\omega^n} + c\frac{d^n\rho_3}{d\omega^n} = 0;$$

whence, if two of the terms are constant, the third is also. Similarly, the n^{th} involute of an epicycloid is defined by $\dfrac{d^n}{d\omega^n}\rho = \text{A}\sin(a\omega + \beta)$. And we see that if two sides envelope such curves, the third will also. The equiangular spiral $\rho = \text{A}e^{\omega}$ has the same property.]

7452. (By G. B. MATHEWS, B.A.)—Prove that (1) if A′, B′, C′ divide the sides BC, CA, AB of the triangle ABC so that BA′ : A′C = CB′ : B′A = AC′ : C′B = $m : n$, the area of the triangle A″B″C″ inclosed by AA′, BB′, CC′ is $(m-n)^2 / (m^2 + mn + n^2)\,\Delta$ ABC, and

(2) B″C″ : AA′ = C″A″ : BB′ = A″B″ : CC′ = $m^2 \sim n^2 : m^2 + mn + n^2$.

Solution by A. MARTIN, B.A.; MARGARET T. MEYER; *and others.*



1. Taking AC, AB as axes, and putting

$$\angle \text{CAB} = \omega,$$

the equations of AA′, BB′, CC′ are

$$y = \frac{mb}{nc}x, \quad \frac{x}{c} + \frac{y(m+n)}{bn} = 1, \quad \frac{x(m+n)}{mc} + \frac{y}{b} = 1;$$

whence the coordinates of A″, B″, C″ are found to be

$$x=\frac{cn^2}{n^2+mn+m^2},\quad y=\frac{mnb}{n^2+mn+m^2};\quad x=\frac{m^2c}{m^2+mn+n^2},\quad y=\frac{n^2b}{m^2+mn+n^2};$$

$$x=\frac{cmn}{m^2+mn+n^2},\quad y=\frac{m^2b}{m^2+mn+n^2};$$

therefore $$\Delta A''B''C''=\frac{cb}{2(m^2+mn+n^2)}\left\{\frac{n(m^3-n^3)+m(n^3-m^3)}{m^2+mn+n^2}\right\}\sin\omega$$

$$=\frac{bc\,(m-n)^2}{2(m^2+mn+n^2)}\sin\omega=\frac{(m-n)^2}{m^2+mn+n^2}\Delta ABC.$$

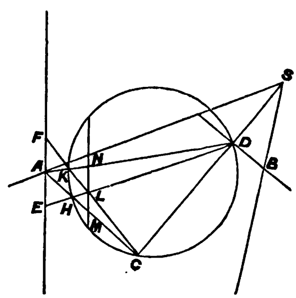
2. Knowing the coordinates of the various points, the second part immediately follows, since, for instance,

$$CC'=b^2+\frac{c^2m^2}{(m+n)^2}+\frac{2bcm}{(m+n)}\cos\omega$$

$$=\frac{1}{(m+n)^2}\left[b^2(m+n)^2+c^2m^2+2(m+n)\,bcm\cos\omega\right].$$

7265. (By J. MACLEOD, M.A.)—In a horizontal plane containing a range AB, a point S is found, in a path parallel to the range, at which the report of a rifle and the sound of the bullet hitting the target are heard simultaneously. SC is the bisector of ASB; and AC, BD are perpendiculars from A, B on SC. EAF is perpendicular to AB; and DE, parallel to SA, meets EF in E. AF is equal to AE, and CF is drawn. Prove that the intersections AC, SC; BD, SC; DE, AC; FC, DA, lie all on the same circle.

Solution by the PROPOSER.



The point in the parallel path is the intersection of the path by a hyperbola whose foci are the extremities of the range, and the common difference of the focal radii equal to the range multiplied by the ratio of the velocity of sound to that of the bullet. By a well-known property of the hyperbola, the points D and C lie on the circle whose diameter is the transverse axis of the hyperbola; also the line through D parallel to SA passes through the centre of the circle, and must therefore intersect CA on the circumference. Let DA intersect the circle in K, and join K, which produce to meet EA in some point F.

By a well-known property of conics, the polar of A passes through L, and is parallel to EF. Also DA : DN = AK : KN = AF : NL = AE : NL, therefore AF = AE. Hence CF intersects DA on the circumference.

6983. (By Professor HADAMARD.) — Si m et n sont deux nombres entiers dont la somme, augmentée de 1, donne un nombre premier, on a

$$m\,!\,n\,! = \text{M . de}\,(m+n+1) \pm 1.$$

Solution by the PROPOSER.

En effet, soit $m+n+1 = p$. Le théorème de Wilson donne:

$$(p-1)\,! = \text{M . de}\,p-1\,;\ \text{mais}\,(p-1)\,! = (p-2)\,!\,(p-1),$$

et évidemment $$p-1 = \text{M . de}\,p-1.$$

En retranchant membre à membre ces deux égalités, on obtient:

$$[(p-2)\,!-1]\,(p-1) = \text{M . de}\,p.$$

Mais p est premier avec $p-1$. Donc $(p-2)\,! = \text{M.de}\,p+1$. Multiplions cette égalité par 2 et ajoutons-lui l'égalité $p-2 = \text{M . de}\,p-2$, en remarquant que $(p-2)\,! = (p-3)\,!\,(p-2)$. Nous avons: $2\,(p-3)\,! = \text{M . de}\,p-1$. De même remplaçons $(p-3)\,!$ par $(p-4)\,!\,(p-3)$; multiplions par 3 et retranchons $p-3 = \text{M.de}\,p-3$; nous avons: $2\,.\,3\,(p-4)\,! = \text{M.de}\,p+1$;

et en général: $$(k-1)\,!\,(p-k)\,! = \text{M . de}\,p \pm 1.$$

Pour démontrer cette loi, comme elle a été vérifiée pour $k=2$, $k=3$, $k=4$, il suffit de montrer que, si elle existe pour une valeur de k, elle existe pour la valeur immédiatement supérieure. Soit donc

$$(k-1)\,!\,(p-k)\,! = \text{M . de}\,p \pm 1,$$

ou $$(k-1)\,!\,[p-(k+1)]\,!\,(p-k) = \text{M . de}\,p \pm 1.$$

Multiplions par k et ajoutons ou retranchons $p-k = \text{M . de}\,p-k$. Nous avons

$(k-1)\,!\,k\,...$, ou $\{k\,[(k+1)-1]\,!\,[p-(k+1)]\,! \pm 1\}\,(p-k) = \text{M . de}\,p$,

ou, puisque p est premier avec $(p-k)$:

$$[(k+1)-1]\,!\,[p-(k+1)]\,! = \text{M.de}\,p \mp 1,$$

ce qui démontre la loi.

Pour $k = p-m = n+1$, la formule devient $n\,!\,m\,! = \text{M . de}\,p \pm 1$.

6737. (By Professor TOWNSEND, F.R.S.)—In the irrotational strain of an incompressible substance in a tridimensional space, if the equipotential surfaces of the strain be a system of confocal ellipsoids in the space, determine the form of the potential ϕ of the strain as a function of the parameter λ of the system.

Solution by the PROPOSER.

The circumstance of the incompressibility of the substance, from which it follows that for every displacement filament of the strain the product of the displacement into the transverse area is constant throughout the entire extent of the filament, enables us to determine the required form very readily as follows.

Denoting by S any ellipsoid of the confocal system, by a, b, c its three semi-axes, by dS a differential element of its area at any point P of its surface, and by p the perpendicular from its centre O on its tangent plane at P; then since, by virtue of the property referred to, the product $\frac{d\phi}{dp} \cdot dS$ is constant throughout the entire extent of the displacement filament of the mass determined by dS, and since, by the geometry of the ellipsoid, $\frac{d\phi}{dp} = \frac{d\phi}{d\lambda} \cdot \frac{d\lambda}{dp} = \frac{d\phi}{d\lambda} \cdot \frac{p}{\lambda}$, therefore the product $\frac{d\phi}{\lambda\, d\lambda} \cdot p\, dS$ is constant throughout the entire extent of the filament; but, by the known properties of corresponding elements of confocal ellipsoids, the product $p\, dS$ throughout the entire extent of the filament is to the product abc in a constant ratio depending on the thickness of the filament, therefore the product $\frac{d\phi}{\lambda\, d\lambda} \cdot abc$ is constant throughout the entire mass; hence, denoting its constant value by k, we have

$$d\phi = k\,[abc]^{-1}\,\lambda\, d\lambda, \;= k\left[(a_0^2+\lambda^2)(b_0^2+\lambda^2)(c_0^2+\lambda^2)\right]^{-\frac{1}{2}}\lambda\, d\lambda,$$

where a_0, b_0, c_0 are the semi-axes of the particular ellipsoid of the system for which $\lambda = 0$, and therefore, observing that its value evidently $= 0$ at infinity,

$$\phi = -k\int_{\lambda}^{\infty}\left[(a_0^2+\lambda^2)(b_0^2+\lambda^2)(c_0^2+\lambda^2)\right]^{-\frac{1}{2}}\lambda\, d\lambda\,;$$

a form identical, as it ought, with that of the attraction potential, for the ordinary law of the inverse square of the distance, of a thin uniform ellipsoidal shell of semi-axes a_0, b_0, c_0, and mass k, the external equipotential surfaces of which are, as is well known, the external ellipsoids confocal with its surface, and therefore identical with those of the supposed irrotational strain of the question.

From the above we see that, if a thin uniform ellipsoidal shell of any expansible substance, holding in free equilibrium by its attraction for the ordinary law of the inverse square of the distance a surrounding mass of any incompressible liquid, receive a small expansion of volume deforming it into a confocal shell, and put in consequence the surrounding liquid into irrotational strain; the displacement lines of the resulting strain will be the intersections of the two systems of hyperboloids of opposite squares confocal with either shell.

5591. (By D. EDWARDES.)—If r be the inscribed radius and s the semiperimeter of a triangle, prove that $s^2 < 27r^2$.

Solution by Professor MOREL; J. O'REGAN; *and others.*

Since $(a-b)^2+(b-c)^2+(c-a)^2$ is positive, $a^2+b^2+c^2 > ab+bc+ca$; hence, if a, b, c be the sides of a triangle, we have $s^2 > 3r^2+12Rr$. But the expression for the distance between the centres of inscribed and circumscribed circles shows that $R > 2r$; hence $s^2 > 3r^2+24r^2 > 27r^2$.

Similarly we can show that $27R^2 > 4s^2$; hence $27R^2 > 4s^2 > 108r^2$.

7298. (By Captain MacMahon, R.A.)—Verify that the equation

$$(A+3Bx+3Cx^2+Dx^3)(A+3By+3Cy^2+Dy^3)(A+3Bz+3Cz^2+Dz^3)$$
$$=[A+B(x+y+z)+C(yz+zx+xy)+Dxyz]$$

leads to the differential relation

$$\frac{dx}{(A+3Bx+3Cx^2+Dx^3)^{\frac{2}{3}}}+\frac{dy}{(\ldots)^{\frac{2}{3}}}+\frac{dz}{(A+3Bz+3Cz^2+Dz^3)^{\frac{2}{3}}}=0.$$

I. *Solution by the* Proposer.

Writing the equation $XYZ = P^3$, and differentiating logarithmically, we have

$$\frac{1}{X}\frac{dX}{dx}dx+\frac{1}{Y}\frac{dY}{dy}dy+\frac{1}{Z}\frac{dZ}{dz}dz=\frac{3}{P}\left\{\frac{dP}{dx}dx+\frac{dP}{dy}dy+\frac{dP}{dz}dz\right\},$$

or $$\left(\tfrac{1}{3}P\frac{dX}{dx}-X\frac{dP}{dx}\right)dx+\left(\tfrac{1}{3}P\frac{dY}{dy}-Y\frac{dP}{dy}\right)dy+\left(\tfrac{1}{3}P\frac{dZ}{dz}-Z\frac{dP}{dz}\right)dz=0.$$

Now, if $X^{-\frac{2}{3}}dx+Y^{-\frac{2}{3}}dy+Z^{-\frac{2}{3}}dz=0$, we should have

$$Y\left\{\tfrac{1}{3}P\frac{dX}{dx}-X\frac{dP}{dx}\right\}^3-X\left\{\tfrac{1}{3}P\frac{dY}{dy}-Y\frac{dP}{dy}\right\}^3=0;$$

expanding the left-hand side, and multiplying out, it is found to be

$$(XYZ-P^3)(x-y)\Big[3(B^2-AC)^2-3(B^2-AC)(AD-BC)(x+y)$$
$$+\left\{(AD-BC^2)-(B^2-AC)(C^2-BD)\right\}(x^2+y^2)+\left\{(AD-BC)^2\right.$$
$$\left.+8(B^2-AC)(C^2-BD)\right\}xy-3(AD-BC)(C^2-BD)xy(x+y)$$
$$+3(C^2-BD)^2x^2y^2\Big],$$

which vanishes, and completes the verification.

II. *Solution by* G. B. Mathews, B.A.

Consider the curve

$$f\equiv y^3-(Ax^3+3Bx^2+3Cx+D)=0.$$

Corresponding to this, we have the Abelian function

$$\int\frac{dx}{f'(y)}=\int\frac{dx}{3y^2},\quad\text{or}\quad\int\frac{dx}{y^2}=\int\frac{dx}{Ax^3+\ldots)^{\frac{2}{3}}}.$$

The differential relation is

$$\frac{dx_1}{(Ax_1^3+\ldots)^{\frac{2}{3}}}+\frac{dx_2}{(Ax_2^3+\ldots)^{\frac{2}{3}}}+\frac{3}{Ax_3^3+\ldots)}=0,$$

and corresponding to this we have the integral relation expressing that the points (1, 2, 3) lie in a line, viz.,

$$(x_2-x_3)y_1+(x_3-x_1)y_2+(x_1-x_2)y_3=0,$$

or $$(x_2-x_3)(Ax_1^3+\ldots)^{\frac{1}{3}}+(x_3-x_1)(Ax_2^3+\ldots)^{\frac{1}{3}}+\ldots=0.$$

This is of the form $\quad P^{\frac{1}{3}}+Q^{\frac{1}{3}}+R^{\frac{1}{3}}=0,$

or, rationalized, $\quad (P+Q+R)^3=27PQR$ ....(*a*),

where, writing x, y, z for x_1, x_2, x_3, we have

$$P = (y-z)^3(Ax^3+3Bx^2+3Cx+D),$$
$$Q = (z-x)^3(Ay^3+\ldots),\quad R = (x-y)^3(Az^3+\ldots).$$

Now $$x(y-z)+\ldots+\ldots \equiv 0,$$

therefore $$x^3(y-z)^3+\ldots+\ldots = 3xyz(y-z)(z-x)(x-y).$$

Again, $$x^2(y-z)+\ldots+\ldots \equiv x^2[y^3-z^3-3yz(y-z)]+\ldots$$
$$\equiv x^2(y^3-z^3)+\ldots+\ldots-3xyz[x(y-z)+\ldots+\ldots]$$
$$\equiv (y-z)(z-x)(x-y)(yz+zx+xy),$$
$$x(y-z)^3+\ldots \equiv (y-z)(z-x)(x-y)(x+y+z),$$
$$(y-z)^3+\ldots \equiv 3(y-z)(z-x)(x-y),$$

therefore $$P+Q+R \equiv 3(y-z)(z-x)(x-y)[Axyz+B(yz+zx+xy)+C(x+y+z)+D],$$

and $$PQR \equiv (y-z)^3(z-x)^3(x-y)^3(Ax^3+3Bx^2+\ldots)(Ay^3+\ldots)(Az^3+\ldots),$$

Substituting in (α) and dividing out $27(y-z)^3(z-x)^3(x-y)^3$, we get

$$[Axyz+B(yz+zx+xy)+C(x+y+z)+D]^3$$
$$=(Ax^3+3Bx^2+3Cx+D)(Ay^3+3By^2+3Cy+D)(Az^3+3Bz^2+3Cz+D).$$

[Mr. MATHEWS states that for the method employed he is principally indebted to Professor CAYLEY's Lectures on "Abelian Functions."]

7352. (By Professor CAYLEY, F.R.S.)—Denoting by $x, y, z, \xi, \eta, \zeta$ homogeneous linear functions of four coordinates, such that identically

$$x+y+z+\xi+\eta+\zeta = 0,\quad ax+by+cz+f\xi+g\eta+h\zeta = 0,$$

where $af = bg = ch = 1$; show that $\sqrt{(x\xi)}+\sqrt{(y\eta)}+\sqrt{(z\zeta)} = 0$ is the equation of a quartic surface having the sixteen singular tangent planes (each touching it along a conic)

$$x=0,\quad y=0,\quad z=0,\quad \xi=0,\quad \eta=0,\quad \zeta=0,$$
$$x+y+z=0,\quad x+\eta+z=0,\quad ax+by+cz=0,\quad ax+g\eta+cz=0,$$
$$\xi+y+z=0,\quad x+y+\zeta=0,\quad f\xi+by+cz=0,\quad ax+by+h\zeta=0,$$
$$\frac{x}{1-bc}+\frac{y}{1-ca}+\frac{z}{1-ab}=0,\quad \frac{\xi}{1-gh}+\frac{\eta}{1-hf}+\frac{\zeta}{1-fg}=0.$$

Solution by W. J. CURRAN SHARP, M.A.

If $$\sqrt{(x\xi)}+\sqrt{(y\eta)}+\sqrt{(z\zeta)} = 0,$$

we have $$2xy\xi\eta+2yz\eta\zeta+2zx\zeta\xi-x^2\xi^2-y^2\eta^2-z^2\zeta^2 = 0 \ldots\ldots\ldots\ldots\ldots(1),$$

a quartic surface. If $x=0$, $(y\eta-z\zeta)^2=0$, or the plane $x=0$ touches along the conic $y\eta-z\zeta=0$; and similarly for the planes $y=0$, $z=0$, $\xi=0$, $\eta=0$, $\zeta=0$.

Again, if $x+y+z=0$, by the first equation of condition $\xi+\eta+\zeta=0$, therefore $x\xi=(y+z)(\eta+\zeta)$, and the equation (1) reduces to

$$[(y+z)(\eta+\zeta)-(y\eta+z\zeta)]^2 = 4yz\eta\zeta \quad\text{or}\quad (z\eta-y\zeta)^2 = 0,$$

and the plane touches along the conic $x+y+z=0$, $z\eta-y\zeta=0$; and similarly for the next three planes. Again, if $ax+by+cz=0$, by the second identity $f\xi+g\eta+h\zeta=0$, and $x\xi=afx\xi=(by+cz)(g\eta+h\zeta)$, and (1) reduces to $[(by+cz)(g\eta+h\zeta)-(y\eta+z\zeta)]^2=4yz\eta\zeta$, or $(cgz\eta-bhy\zeta)^2=0$ by the given conditions; and therefore the plane $ax+by+cz=0$ touches (1) along the conic $ax+by+cz=0$, $cgz\eta-bhy\zeta=0$; and similarly the planes $ax+g\eta+cz=0$, $f\xi+by+cz=0$, $ax+by+h\zeta=0$ touch along conics, the equations to which may be derived from the above.

If $\dfrac{x}{1-bc}+\dfrac{y}{1-ca}+\dfrac{z}{1-ab}=0$, it is easy to show that

$$x:y:z=(b-c)(1-bc)\left\{-(a-b)\frac{\eta}{b}+(c-a)\frac{\zeta}{c}\right\}$$

$$:(c-a)(1-ca)\left\{-(b-c)\frac{\zeta}{c}+(a-b)\frac{\xi}{a}\right\}$$

$$:(a-b)(1-ab)\left\{-(c-a)\frac{\xi}{a}+(b-c)\frac{\eta}{b}\right\};$$

and therefore

$$x\xi:y\eta:z\zeta=a(b-c)(1-bc)\left\{-(a-b)\frac{\xi\eta}{ab}+(c-a)\frac{\zeta\xi}{ca}\right\}$$

$$:b(c-a)(1-ca)\left\{-(b-c)\frac{\eta\zeta}{bc}+(a-b)\frac{\xi\eta}{ab}\right\}$$

$$:c(a-b)(1-ab)\left\{-(c-a)\frac{\zeta\xi}{ca}+(b-c)\frac{\eta\zeta}{bc}\right\},$$

or $\quad A(-Z+Y):B(-X+Z):C(-Y+X)$, where $A+B+C=0$;

therefore the equation to the surface is

$$[A(Y-Z)]^{\frac{1}{2}}+[B(Z-X)]^{\frac{1}{2}}+[C(X-Y)]^{\frac{1}{2}}=0,$$

therefore $\quad A(Y-Z)+B(Z-X)+2[AB(Y-Z)(Z-X)]^{\frac{1}{2}}=C(X-Y)$;

$\therefore \quad -(B+C)X+(A+C)Y-(A-B)Z+2[AB(Y-Z)(Z-X)]^{\frac{1}{2}}=0,$

or $\quad A(X-Z)+B(Z-Y)+2[AB(Y-Z)(Z-X)]^{\frac{1}{2}}=0,$

and the plane touches along

$$[A(X-Z)]^{\frac{1}{2}}+[B(Z-Y)]^{\frac{1}{2}}=0 \quad\text{or}\quad AX+BY+CZ=0;$$

that is, $\quad a(b-c)(1-bc)\dfrac{\eta\zeta}{bc}+b(c-a)(1-ca)\dfrac{\zeta\xi}{ca}+c(a-b)(1-ab)\dfrac{\xi\eta}{ab}=0,$

a conic. And similarly in the case of the last plane.

[Mr. Sharp remarks that he does not see why Professor Cayley has not reckoned the six planes $\dfrac{x}{1-cg}+\dfrac{\eta}{1-ca}+\dfrac{\ }{1-ag}=0,$

$$\frac{x}{1-gh}+\frac{\eta}{1-ah}+\frac{\zeta}{1-ag}=0,\qquad \frac{\xi}{1-bc}+\frac{y}{1-fc}+\frac{z}{1-bf}=0,$$

$$\frac{\xi}{1-gc}+\frac{\eta}{1-fc}+\frac{z}{1-fg}=0,\qquad \frac{x}{1-bh}+\frac{\eta}{1-ah}+\frac{\zeta}{1-ac}=0,$$

$\dfrac{\xi}{1-bh}+\dfrac{y}{1-fh}+\dfrac{\zeta}{1-bf}=0$, as tangent planes of the same kind.]

7454. (By Professor SYLVESTER, F.R.S.)—If I, an invariant of the i^{th} order of $(a_0, a_1, a_2 \ldots)(x, y)^m$, becomes I′ when, for any suffix θ, a_θ becomes $a_{\theta+1}$, prove that $\quad \mathrm{I} = \phi \mathrm{I}'$, where $\phi = \Sigma \dfrac{\mathrm{E}_r^\lambda \cdot \mathrm{E}_s^\mu \cdot \mathrm{E}_t^\nu \ldots}{\lambda \cdot \pi\mu \cdot \pi\nu \ldots}$,
E in general signifying

$$a_0 \frac{d}{da_\epsilon} + \epsilon a_1 \frac{d}{da_{\epsilon+1}} + \frac{\epsilon(\epsilon+1)}{1 \cdot 2} a_2 \frac{d}{da_{\epsilon+2}} + \frac{\epsilon(\epsilon+1)(\epsilon+2)}{1 \cdot 2 \cdot 3} a_3 \frac{d}{da_{\epsilon+3}} + \ldots,$$

and $\lambda, \mu, \nu \ldots r, s, t \ldots$ being any positive integers satisfying the condition $\lambda r + \mu s + \nu t + \ldots = i$.

Solution by ROBERT RUSSELL, B.A.

Let $$u = (a_0, a_1, a_2 \ldots)(xy)^n,$$

$$\mathrm{U} = (a_0, a_1, a_2 \ldots)(x + \theta y, y)^n \; (\mathrm{A}_0 \mathrm{A}_1 \ldots)(x, y)^n, \quad \mathrm{V} = (a_1 a_2 \ldots)(x, y)^n;$$

then, if we form the I-invariant of $\mathrm{V} + \theta\mathrm{U}$, the coefficient θ^i will be the original I-invariant of u,

$$\mathrm{I}(\mathrm{V} + \theta\mathrm{U}) = \left(1 + \theta\delta + \frac{\theta^2\delta^2}{2!} + \frac{\theta^3\delta^3}{3!} + \ldots\right) \mathrm{I}' = e^{\theta\delta} \cdot \mathrm{I}',$$

where $$\delta = \mathrm{A}_0 \frac{d}{da_1} + \mathrm{A}_1 \frac{d}{da_2} + \mathrm{A}_2 \frac{d}{da_3} + \ldots = \mathrm{E}_1 + \theta \mathrm{E}_2 + \theta^2 \mathrm{E}_3 + \ldots$$

Hence I = coefficient of θ^i in

$$e^{\theta \mathrm{E}_1 + \theta^2 \mathrm{E}_2 + \theta^3 \mathrm{E}_3 + \ldots} \mathrm{I}' = \Sigma \frac{\mathrm{E}_r^\lambda \mathrm{E}_s^\mu \mathrm{E}_t^\nu \ldots}{\lambda!\, \mu!\, \nu! \ldots} \mathrm{I}', \text{ where } \lambda r + \mu s + \nu t \ldots = i.$$

7484. (By Professor MALET, F.R.S.)—If two solutions of the linear differential equation (A) are the solutions of the equation (B),

$$\frac{d^3y}{dx^3} + \mathrm{Q}_1 \frac{d^2y}{dx^2} + \mathrm{Q}_2 \frac{dy}{dx} + \mathrm{Q}_3 y = 0, \quad \frac{d^2y}{dx^2} + \mathrm{P}_1 \frac{dy}{dx} + \mathrm{P}_2 y = 0 \ldots (\mathrm{A}, \mathrm{B});$$

prove that (1)

$$\mathrm{P}_1 \mathrm{P}_2 (\mathrm{P}_1 - \mathrm{Q}_1) = \mathrm{P}_2 \left(\frac{d\mathrm{P}_1}{dx} + \mathrm{P}_2 - \mathrm{Q}_2\right) = \mathrm{P}_1 \left(\frac{d\mathrm{P}_2}{dx} - \mathrm{Q}_3\right),$$

and (2) the complete solution of (A) is the solution of

$$\frac{d^2y}{dx^2} + \mathrm{P}_1 \frac{dy}{dx} + \mathrm{P}_2 y = c \mathrm{P}_2 e^{-\int \frac{\mathrm{Q}_3}{\mathrm{P}_2} dx}.$$

Solution by R. RAWSON; G. B. MATHEWS, B.A.; *and others.*

In (A) substitute $y = y_1 \int v\, dx$, where (y_1) is a particular solution of (A),

then $$\frac{d^2v}{dx^2} + \left(3 \frac{dy_1}{y_1 dx} + \mathrm{Q}_1\right) \frac{dv}{dx} + \frac{3}{y_1} \left\{ \frac{d^2y_1}{dx^2} + \frac{2\mathrm{Q}_1}{3} \frac{dy_1}{dx} + \frac{\mathrm{Q}_2}{3} y_1 \right\} v = 0 \;\ldots (3).$$

In (3) substitute $v = v_1 \int w\,dx$, where (v_1) is a particular solution of (3),

then $$\frac{dw}{w\,dx} + 2\frac{dv_1}{v_1\,dx} + 3\frac{dy_1}{y_1\,dx} + Q_1 = 0, \text{ or } w = \frac{e^{-\int Q_1\,dx}}{v_1^2\,y_1^3} \quad \text{.........(4).}$$

Since v_1 and $v_1 \int \frac{e^{-\int Q_1\,dx}}{v_1^2 y_1^3}\,dx$ are each particular solutions of (3), it follows, therefore, that y_1, $y_1\int v_1\,dx$, $y_1\int\left\{v_1\int\frac{e^{-Q_1\,dx}}{v_1^2 y_1^3}\,dx\right\}dx$ are each particular solutions of (A). In (1) substitute $Y_1\int V dx$, where Y_1 is a particular solution of (1), then particular solutions of (1) are given by

$$Y_1,\ Y_1\int\frac{e^{-\int P_1\,dx}}{Y_1^2}\,dx.$$

Instead of (2) take $$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = ce^{\int R\,dx} \quad \text{........................(5),}$$

where R is a function of x, and in (5) substitute $y = Y_1\int U\,dx$, where Y_1 is, of course, a solution of (1); therefore

$$cY_1\int\left\{\frac{e^{-\int P_1\,dx}}{Y_1^2}\int Y_1 e^{\int(P_1+R)\,dx}\,.\,dx\right\}dx \quad \text{..................(6)}$$

is a particular solution of (5). If, therefore, $y_1 = Y_1$, by the question,

$$\frac{d^3y_1}{dx^3} + Q_1\frac{d^2y_1}{dx^2} + Q_2\frac{dy_1}{dx} + Q_3 y_1 = 0, \quad \frac{d^2y_1}{dx^2} + P_1\frac{dy_1}{dx} + P_2 y_1 = 0 \quad \text{......(7).}$$

Differentiate the latter, and from the result take the former; then

$$\frac{d^2y_1}{dx^2} + \frac{\frac{dP_1}{dx} + P_2 - Q_2}{P_1 - Q_1}\frac{dy_1}{dx} + \frac{\frac{dP_2}{dx} - Q_3}{P_1 - Q_1}y_1 = 0 \quad \text{..................(8).}$$

Equation (8) cannot, therefore, be satisfied except

$$\frac{dP_1}{dx} + P_2 - Q_2 = P_1(P_1 - Q_1), \quad \frac{dP_2}{dx} - Q_3 = P_2(P_1 - Q_1) \quad \text{.........(9),}$$

which are the conditions given in the question.

Since $\frac{Q_3}{P_2} = \frac{dP_2}{P_2\,dx} - P_1 + Q_1$, equation (2) readily reduces to

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = ce^{\int(P_1 - Q_1)\,dx} \quad \text{.....................(10),}$$

which coincides with (5) when $R = P_1 - Q_1$.

Again, if the second particular solution of (A) is equal to the second solution of (1), then $$v_1 = \frac{e^{-\int P_1\,dx}}{y_1^2}.$$

This value of (v_1) satisfies (3) by means of the conditions (9). Hence,

the two first particular solutions of (A) are equal to the two particular solutions of (1).

Substitute the value of (v_1) above given in the third particular solution of (A), then it becomes $cy_1 \int \left\{ \frac{e^{-\int P\,dx}}{y_1^2} \int y_1 e^{\int (2P_1 - Q_1)\,dx} dx \right\} dx$,

which coincides with (6) when the proper substitutions, viz., $y_1 = Y_1$, $R = P_1 - Q_1$, are made. Hence, the truth of the proposition.

It follows, therefore, from the conditional equations, that

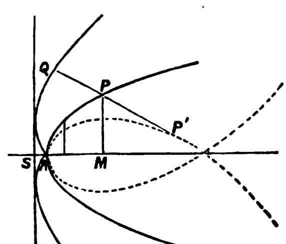
$$\frac{d^3y}{dx^2} + \left\{ P_2 e^{-\int \frac{Q_3}{P_2} dx} \int e^{\int \frac{Q_3}{P_2} dx} \left(\frac{Q_2}{P_2} - 1\right) dx + \frac{Q_3}{P_2} - \frac{dP_2}{P_2\,dx} \right\} \frac{d^2y}{dy^2} + Q_2 \frac{dy}{dx} + Q_3 y = 0$$

has, for its complete solution, the solution of

$$\frac{d^2y}{dx^2} + \left\{ P_2 e^{-\int \frac{Q_3}{P_2} dx} \int e^{\int \frac{Q_3}{P_2} dx} \left(\frac{Q_2}{P_2} - 1\right) dx \right\} \frac{dy}{dx} + P_2 y = c e^{-\int \frac{Q_3}{P_2} dx}.$$

7220. (By Professor WOLSTENHOLME, M.A., D.Sc.)—If S be the given focus and A the given vertex of an ellipse, prove that (1) the straight line joining the second focus to the ends of the minor axis will envelope a curve of degree 4 and class 3, which is the involute starting from the vertex of the first negative pedal (with respect to the focus) of the parabola whose vertex is A, and whose directrix cuts SA at right angles in S; and (2) if Q, P be corresponding points on this curve and on the parabola, and PM be drawn perpendicular to the axis, PM = PQ, so that the circle with centre on the parabola which touches the axis will also envelope this same curve.

Solution by the PROPOSER.



Taking S as origin, axis of x along SA, and SA $= a$, then, if $2m$ be the major axis and e the excentricity of the ellipse, the equation of one of the lines is

$$\frac{x + 2me}{me} = \frac{y}{m(1-e^2)^{\frac{1}{2}}},$$

or, if $e = \frac{\lambda^2 - 1}{\lambda^2 + 1}$, the equation is

$$x\lambda + \tfrac{1}{2}y(1-\lambda^2) - a(1-\lambda^2)\lambda = 0,$$

or $$x - \tfrac{1}{2}y\left(\lambda - \frac{1}{\lambda}\right) = a(1-\lambda^2);$$

whence, for the point of contact with the envelope,

$$\tfrac{1}{2}y\left(1 + \frac{1}{\lambda^2}\right) = 2a\lambda, \quad y = \frac{4a\lambda^3}{1+\lambda^2}, \quad \text{and} \quad x = \frac{a(1-\lambda^2)^2}{1+\lambda^2}.$$

Hence $\qquad \dfrac{dx}{d\lambda} = -\dfrac{2a\lambda(1-\lambda^2)(3+\lambda^2)}{(1+\lambda^2)^2}, \quad \dfrac{dy}{d\lambda} = \dfrac{4a\lambda^2(3+\lambda^2)}{(1+\lambda^2)^2};$

and there are cusps when $\lambda = 0$, or $\pm(-3)^{\frac{1}{2}}$; *i.e.*, at the points $(a, 0)$, $[-8a, \pm 6(-3)^{\frac{1}{2}}a]$; or, moving the origin to A, $(0, 0)$, $[-9a, \pm 6(-3)^{\frac{1}{2}}a]$; and the equations of the joining lines are $2x + y(-3)^{\frac{1}{2}} = 0$, $2x - y(-3)^{\frac{1}{2}} = 0$, $x + 9a = 0$. The values of these three quantities, at the point λ of the curve, are $\dfrac{2a\lambda^2}{1+\lambda^2}[\lambda + (-3)^{\frac{1}{2}}]^2$, $\dfrac{2a\lambda^2}{1+\lambda^2}[\lambda - (-3)^{\frac{1}{2}}]^2$, $\dfrac{a(\lambda^2+3)^2}{1+\lambda^2}$,

so that $[2x + y(-3)^{\frac{1}{2}}]^{-\frac{1}{2}} : [2x - y(-3)^{\frac{1}{2}}]^{-\frac{1}{2}} : (x+9a)^{-\frac{1}{2}}$

$$= [\lambda - (-3)^{\frac{1}{2}}] : \lambda + (-3)^{\frac{1}{2}} : \lambda 2^{\frac{1}{2}},$$

and the equation is therefore

$$[x + \tfrac{1}{2}y(-3)^{\frac{1}{2}}]^{-\frac{1}{2}} + [x - \tfrac{1}{2}y(-3)^{\frac{1}{2}}]^{-\frac{1}{2}} + 2(x+9a)^{-\frac{1}{2}} = 0.$$

The equation of the normal at the point λ is (A origin)

$$\left(x - \frac{a\lambda^2(\lambda^2-3)}{1+\lambda^2}\right)(\lambda^2-1) + \left(y - \frac{4a\lambda^3}{1+\lambda^2}\right)2\lambda = 0,$$

or

$$x(\lambda^2-1) + 2\lambda y = a\lambda^2(\lambda^2+3);$$

and the perpendicular on it from $(a, 0)$ is $(x-a)2\lambda = y(\lambda^2-1)$, hence at the foot of the perpendicular $x = a\lambda^2$, $y = 2a\lambda$, or the point lies on the parabola $y^2 = 4ax$. Thus the normal to the tricusp is a straight line drawn from a point of the parabola at right angles to the focal radius vector, and therefore envelopes the first negative pedal of the parabola with respect to the focus [whose equation is $27ay^2 = x(x-9a)^2$]. The tricusp is therefore an involute of this curve, starting from the vertex, since it has a cusp at that point.

If Q be the point λ on the tricusp, P, P′ the corresponding points on the parabola and its first negative pedal; we have already seen that

$$(1)\ x_1 = \frac{a\lambda^2(\lambda^2-3)}{1+\lambda^2}, \quad y_1 = \frac{4a\lambda^3}{1+\lambda^2}; \qquad (2)\ x_2 = a\lambda^2, \quad y_2 = 2a\lambda;$$

and, by taking the envelope of the normal at Q, we get

$$x\left(1 + \frac{1}{\lambda^2}\right) = 3a(1+\lambda^2), \quad \text{or} \quad x_3 = 3a\lambda^2,$$

and therefore $\qquad y_3 = a\lambda(3-\lambda^2).$

Hence $\qquad x_1 - x_2 = -\dfrac{4a\lambda^2}{1+\lambda^2}, \quad y_1 - y_2 = 2a\lambda\dfrac{\lambda^2-1}{\lambda^2+1},$

so that $PQ^2 = 4a^2\lambda^2$; or $PQ = 2a\lambda = y_2$; and the circle with P as centre and QP as radius will touch the tricusp in Q, and will also touch the axis. Hence also the tangents at P, Q will meet on the axis of x. Hence the tricusp is also the envelope of the circle whose centre is on the parabola, and whose radius is the ordinate to the centre.

If s_1, s_2, s_3 be the arcs of the three curves measured from A,

$$\frac{ds_1}{d\lambda} = 2a\lambda\frac{(3+\lambda^2)}{1+\lambda^2} = 2a\lambda + \frac{4a\lambda}{1+\lambda^2}, \quad s_1 = a[\lambda^2 + 2\log(1+\lambda^2)];$$

$$\frac{ds_2}{d\lambda} = 2a(1+\lambda^2)^{\frac{1}{2}}, \text{ and } s_2 = a\left\{\lambda(1+\lambda^2)^{\frac{1}{2}} + \log[\lambda + (1+\lambda^2)^{\frac{1}{2}}\right\},$$

$$\frac{ds_3}{d\lambda} = 3a(1+\lambda^2), \quad s_3 = a(3\lambda + \lambda^3).$$

The equation of the tricusp is

$$\frac{2x}{x^2+\frac{3}{4}y^2}+\frac{2}{(x^2+\frac{3}{4}y^2)^{\frac{1}{2}}}=\frac{4}{x+9a},$$

or $$(3y^2+2x^2-18ax)^2=(x+9a)^2(4x^2+3y^2);$$

or $$y^4+x^2y^2-2ax(8x^2+9y^2)-27a^2y^2=0;$$

or, solving with respect to y^2,

$$2y^2+x^2-18ax-27a^2=\left[(x^2-18ax-27a^2)^2+64ax^3\right]^{\frac{1}{2}}=\left[(x+a)(x+9a)^3\right]^{\frac{1}{2}},$$

and expanding this in descending powers of x, $=\pm(x^2+14ax+37a^2)+\&c.$ Thus, at infinity, $y^2=16ax+32a^2$, a parabolic asymptote; and $x^2+y^2-4ax+10a^2=0$, a circular asymptote to the impossible branch which passes through the *cyclic* points, and itself altogether impossible. Coordinates of P, Q, P′ are

$$a(1+\lambda^2),\ 2a\lambda;\quad \frac{a(1-\lambda^2)^2}{1+\lambda^2},\ \frac{4a\lambda^3}{1+\lambda^2};\quad a(1+3\lambda^2),\ a\lambda(3-\lambda^2);$$

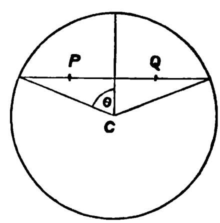
where the excentricity of the ellipse is $\dfrac{\lambda^2-1}{\lambda^2+1}$;

arc $\text{AQ}=a\left[\lambda^2+2\log(1+\lambda^2)\right]$, arc $\text{AP}=a\left\{\lambda(1+\lambda^2)^{\frac{1}{2}}+\log\left[\lambda+(1+\lambda^2)^{\frac{1}{2}}\right]\right\}$

and $$\text{arc AP}'=\text{P}'\text{Q}=a\lambda(3+\lambda^2).$$

7329. (By the late Professor Seitz, M.A.)—Show that the average area of a triangle drawn on the surface of a given circle of radius r, having its base parallel to a given line, and its vertex taken at random, is $\dfrac{256r^2}{525\pi}$.

Solution by D. Edwardes; Professor Roy, M.A.; *and others.*



Draw a chord parallel to the given direction, subtending an angle 2θ at the centre, and let P, Q be two points therein, and $\text{PQ}=z$. Let r be the radius, and p the perpendicular from a random point on the chord. Then, for the value of Σp, an investigation of the areas and positions of the centroids of the segments into which the chord divides the circle, gives easily

$$\Sigma p=r^3\left[\tfrac{4}{3}\sin^3\theta-2\theta\cos\theta+\pi\cos\theta+2\sin\theta\cos^2\theta\right].$$

Again, $$\Sigma\text{PQ}=\int_0^{2r\sin\theta}z\,dz\,(2r\sin\theta-z)=\tfrac{4}{3}r^3\sin^3\theta.$$

Then, Sum of areas $$=\int_0^{\frac{1}{2}\pi}\tfrac{1}{2}\Sigma p\,.\,\Sigma\text{PQ}\,.\,r\sin\theta\,d\theta$$

$$=\tfrac{2}{3}r^7\int_0^{\frac{1}{2}\pi}\left[-\tfrac{2}{3}\sin^7\theta-2\theta\cos\theta\sin^4\theta+\pi\cos\theta\sin^4\theta+2\sin^5\theta\right]d\theta.$$

Also if n be the number of ways the points are taken, so that the triangle

may go through all possible variations, evidently

$$n = \int_0^{\frac{1}{2}\pi} \pi r^2 \,.\, r \sin\theta \, d\theta \,.\, 2r^2 \sin^2\theta = \tfrac{4}{3}\pi r^5\,;$$

hence the required average is

$$\frac{r^2}{\pi}\int_0^{\frac{1}{2}\pi} \left[-\tfrac{1}{3}\sin^7\theta - \theta\cos\theta\sin^4\theta + \tfrac{1}{2}\pi\cos\theta\sin^4\theta + \sin^5\theta\right] d\theta = \frac{256r^2}{525\pi}.$$

7333. (By the Rev. T. P. KIRKMAN, M.A., F.R.S.)—Approaching each other from rest at equal heights in the same normal section of two smooth planes, each making an angle θ with the horizon, slide by gravity two equal smooth spheres of homogeneous matter perfectly rigid and incompressible. About the lowest points p and q of their paths, the planes are scooped spherically in their inferior surface, so that the thickness at p and q vanishes. At the instant t of collision, two other spheres exactly like the former impinge by projection from below perpendicularly on the planes at the points p and q, with the same velocity $v \tan\theta$, v being the velocity acquired by the descending spheres. Required, for the peace of mind of Dr. MUSTBESO, an orthodox account of the motion.

Solution by the PROPOSER.

Resolving along vertical and horizontal axes, the velocities of the four spheres, we obtain for each, at the time t, $v = 0$ in all directions; *i.e.*, the two descending spheres remain at rest at the points p and q, while the two others begin to descend at the time t from rest at those points by gravity. But $\Sigma mv^2 > 0$ being the sum of four positive quantities; how is this *vis viva* conserved? Not by thermal or other insensible vibrations of the spheres; for by definition they cannot be compressed and made to recoil or vibrate in any way, so as to communicate vibrations to any medium. Not by vibrations in the planes; for at the time t the planes receive no blow. The *vis viva* seems to be destroyed at the time t without compensation of any kind.

If it be denied that there are in the cosmos any perfectly rigid and incompressible spheres, what becomes of the atomic constitution of matter? If it be affirmed that particles of matter cannot actually collide, by reason of a repulsion working at close quarters by *actio in distans* between those points without parts, their centres of gravity; I remark that, if the inscrutable Cause were to act in the same manner as at present in the lines connecting those ever-moving centres, in the absence of all matter, the facts of the universe would be to us exactly what they are now. Who can prove the presence of this wonderful matter? What clear question about the *data* of the cosmos given to us all is answered by the word *matter?* If any one informs me that he has a concept of matter, I reply that unfortunately I have none, but that, if he will kindly produce his concept, I will try to study it and to become wiser. As to the m for mass in Dynamics, what is it but a number determined by experiment, which would yield the same answer on the above supposition in the absence of matter? Is it enough to reply to these questions—that's all Metaphysics? Quite.

[To Mr. KIRKMAN's query, "How is this *vis viva* conserved?" Mr. CARR remarks that "the answer is contained in the formula $\infty \times 0 =$ finite magnitude. An increasing number of vibrations of diminishing amplitude fulfil the conditions, in the limit if an incompressible body (non-existent in fact), and give a thermal equivalent of Σmv^2. Why four spheres and two planes? Two spheres impinging in the same right line would illustrate the same thing."]

7057. (By J. GRIFFITHS, M.A.)—If

$$\cos\phi\cos\psi + \left(\frac{1-mnk^4}{1+mnk^2}\right)^{\frac{1}{2}}\sin\phi\sin\psi = \left(\frac{1-mn}{1+mnk^2}\right)^{\frac{1}{2}}.\,k\sin\phi;$$

and $\dfrac{1+k^2}{1+mnk^2} = \dfrac{2}{m+n}$, show (1) that

$$\frac{d\psi}{(1+mk\sin\psi\,.\,1-nk\sin\psi)^{\frac{1}{2}}} = \left(\frac{1+k^2}{1+mnk^2}\right)^{\frac{1}{2}}.\,\frac{d\phi}{(1-k^2\sin^2\phi)^{\frac{1}{2}}};$$

and (2) deduce LANDEN's transformation. [Take $k<1$ and $mn<1$.]

Solution by D. EDWARDES; *the* PROPOSER; *and others.*

1. The relation $\dfrac{1+k^2}{1+mnk^2} = \dfrac{2}{m+n}$ gives

$$2\,(1-mn)^{\frac{1}{2}}(1-mnk^4)^{\frac{1}{2}} = (m-n)\,(1+k^2);$$

hence the equation connecting the amplitudes gives

$$\cos^2\phi = \frac{[k\,(1-mn)^{\frac{1}{2}}-(1-mnk^4)^{\frac{1}{2}}\sin\psi]^2}{(1+k^2)\,[1-nk\sin\psi\,.\,1+mk\sin\psi]},$$

also
$$\tan\phi = \frac{(1+mnk^2)^{\frac{1}{2}}\cos\psi}{k\,(1-mn)^{\frac{1}{2}}-(1-mnk^4)^{\frac{1}{2}}\sin\psi};$$

whence, differentiating this last, we have

$$\frac{d\phi}{d\psi} = \left(\frac{1+mnk^2}{1+k^2}\right)^{\frac{1}{2}}\frac{(1-k^2\sin^2\phi)^{\frac{1}{2}}}{[1-nk\sin\psi\,.\,1+mk\sin\psi]^{\frac{1}{2}}},$$

the stated result.

2. Writing $-\phi$ for ϕ, and putting $m=0$, and making the transformation $\sin\psi = \dfrac{(1+k)^2-2\,(1+k^2)\sin^2\phi_1}{(1+k)^2-4k\sin^2\phi_1}$, we have, since $n = \dfrac{2}{1+k^2}$,

$$\frac{d\psi}{(1-nk\sin\psi)^{\frac{1}{2}}} = -\frac{2\,(1+k^2)^{\frac{1}{2}}\,d\phi_1}{[(1+k)^2-4k\sin^2\phi_1]^{\frac{1}{2}}}.$$

Also $\cos\psi = \dfrac{2\,(1-k)\sin\phi_1\cos\phi_1}{(1+k)\,(1-c^2\sin^2\phi_1)}$, where $c^2 = \dfrac{4k}{1+k}$, and the equation connecting the amplitudes becomes $\sin(2\phi_1-\phi) = k\sin\phi$, and we have LANDEN's transformation, viz., $F(c,\phi_1) = \frac{1}{2}(1+k)\,F(k,\phi)$.

[If $m = 0$, $n = \frac{2}{1+k^2}$, the integral equation is $\cos(\psi-\phi) = k\sin\phi$, and we have, from the stated result,

$$\frac{d\psi}{(1+k^2-2k\sin\psi)^{\frac{1}{2}}} = \frac{d\phi}{(1-k^2\sin^2\phi)^{\frac{1}{2}}}; \text{ or, if } \psi = 2\phi_1 - \tfrac{1}{2}\pi,$$

$$\frac{2d\phi_1}{[(1+k)^2-4k\sin^2\phi_1)]^{\frac{1}{2}}} = \frac{d\phi}{(1-k^2\sin^2\phi)^{\frac{1}{2}}}, \text{ where } \sin(2\phi_1-\phi) = k\sin\phi.$$

This is LANDEN's equation.]

7433 & 7443. (By the EDITOR.)—Show that the volume of the greatest parcel that can be sent by the Parcel Post is (1) $8/\pi = 2{\cdot}5468$ ft. when unlimited in form and therefore a right circular cylinder, and (2) 2 cubic feet when it is to be four-sided and plane.

Solution by W. M. MEE, B.A.; Professor ROY; *and others.*

1. The parcel must clearly be a right circular cylinder; thus, putting x for the length of its axis, and r for the radius of its base, we have length + girth $= 6$ ft. $= x + 2\pi y$, and volume $= \pi r^2 x =$ a maximum. These equations readily give $x =$ length $= 2$ ft., girth $= 2\pi y = 4$ ft., radius $= 2/\pi$ ft., and volume $= \pi xy^2 = 8/\pi$ cubic feet.

2. Here the parcel must evidently be a square parallelepiped, and in like manner to (1) it appears at once that its length must be 2 feet, each of its sides 1 foot, its girth 4 feet, and its volume 2 cubic feet.

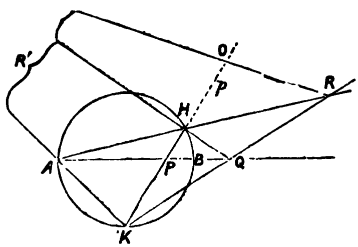
7468. (By S. TEBAY, B.A.)—Find an integral value of a, such that 101^2+a and 101^2-a shall be rational squares.

Solution by G. HEPPEL, M.A.; W. G. LAX, B.A.; *and others.*

Let $$101^2+a = (101+k)^2, \quad 101^2-a = (101-l)^2,$$
$$a = k(202+k) = l(202-l);$$
hence l is greater than k; thus, putting $l = k+c$, we have
$$k(202+k) = (k+c)(202-k-c), \quad 2k^2 = 202c-2kc-c^2.$$
Now, if c is a measure of k, the only values are 2 and 101; and neither of these lead to a solution. But if c does not measure k, it must measure $2k^2$, and hence the only other possible value of c is 4. This gives $k^2+4k-396 = 0$, $(k+22)(k-18) = 0$; so that $k = 18$, $a = 3960$; and $101^2+a = 119^2$, $101^2-a = 79^2$.

7479. (By R. TUCKER, M.A.)—P, Q are lines parallel to the directrix of a parabola; from any point p on P tangents are drawn to the curve cutting Q in r, s; through r, s lines are drawn parallel to the tangents, and meeting in t: prove that these lines envelope a parabola, and that pt passes through the pole of P.

Solution by G. B. MATHEWS, B.A.; R. KNOWLES, B.A., L.C.P.; *and others.*



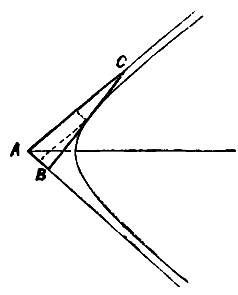
The theorem can be easily proved by reciprocation with regard to the focus of the parabola, the reciprocal theorem being as follows:—

P, Q are any two points on the diameter AB of a given circle; HPK is a chord through P; AH, QK meet in R, and AK, QH in R′; then the locus of R, R′ is one and the same hyperbola which passes through A.

The chord HK corresponds to the point p and the line RR′ to the point t; so that the second part of the proposition amounts to showing that the point O where RR′, HK intersect lies on the polar of P, which is obvious, since KPHO is a harmonic range.

7505. (By G. HEPPEL, M.A.)—If three hyperbolas be described, to each of which one side of a given triangle is a tangent, and the other sides are asymptotes, show that the product of the three latera recta is equal to the cube of the diameter of the inscribed circle.

Solution by R. KNOWLES, B.A., L.C.P.; W. G. LAX, B.A.; *and others.*



Let α, β be the semiaxes, and e the eccentricity of the hyperbola opposite A; then, if a, b, c, A, B, C refer to the fixed triangle,

$$\tfrac{1}{4}bc = \tfrac{1}{4}(\alpha^2+\beta^2), \quad \text{or} \quad bc = \alpha^2 e^2,$$

and
$$\cos\tfrac{1}{2}A = \frac{1}{e},$$

therefore
$$l_1 = \alpha(e^2-1) = \left(e-\frac{1}{e}\right)\alpha e$$

$$= (bc)^{\frac{1}{2}}\frac{\sin^2\frac{1}{2}A}{\cos\frac{1}{2}A} = \frac{s_2 s_3}{(ss_1)},$$

where $s_1 = s-a$, &c.; hence we have

$$l_1 l_2 l_3 = \frac{(s s_1 s_2 s_3)^{\frac{3}{2}}}{s^3} = \frac{\Delta^3}{s^3} = r^3, \text{ therefore } 2l_1 \,.\, 2l_2 \,.\, 2l_3 = (2r)^3.$$

7496. (By R. A. Roberts, M.A.)—A geodesic common tangent is drawn to two circular sections of an ellipsoid; show (1) that the perpendiculars from the centre on the tangent planes to the surface at the points of contact are equal; and hence (2) find the locus of the points of contact of the geodesic tangents drawn from an umbilic to the circular sections.

Solution by T. Woodcock, B.A.; Professor Nash, M.A.; *and others.*

1. Along any geodesic on the ellipsoid, pD = constant, p being the perpendicular upon the tangent plane at any point P, and D the semi-diameter parallel to the geodesic's tangent at P. Let p', D' be the lengths of the same lines for any other point P′, and let a, b, c be the semi-axes of the ellipsoid. If the same geodesic touch one circular section at P and another at P′, we have $D = b = D'$; therefore $p = p'$.

2. Along all geodesics through an umbilic, $pD = ac$. Therefore, at the points of contact of these geodesics with the circular sections $p = \dfrac{ac}{b}$. This equation represents a polhode on the surface of the ellipsoid.

7245. (By R. Knowles, B.A., L.C.P.)—Three normals are drawn from a point to a parabola, and tangents are then drawn at the points where the normals meet the curve; prove (1) that the area of the triangle formed by the tangents is *half* that formed by joining the points in the curve; (2) if the point moves on a given straight line, the locus of each of its angular points is the same hyperbola.

Solution by W. H. Blythe, M.A.; Professor Matz, M.A.; *and others.*

1. This applies to any three points (m_1, m_2, m_3) on the parabola, $y^2 = 4ax$, putting the coordinates into the form $x = am^2$, $y = 2am$. Then the points of intersection of tangents become am_1m_2, $a(m_1+m_2)$, &c.; taking ordinary formulæ for the area of a triangle. First, for the three points $(am_1^2 . 2am_1)$, &c., we get $a^2(m_1-m_2)(m_2-m_3)(m_3-m_1)$; next, taking the points am_1m_2, $a(m_1+m_2)$, &c., we get

$$\tfrac{1}{2}a^2(m_1-m_2)(m_2-m_3)(m_3-m_1).$$

2. We take m_1, m_2, m_3 as the roots of the equation

$$am^3 + m(h-2a) + k = 0,$$

(hk) being the point at which the normals meet, and, since (hk) moves on a fixed line, we may take $k = bh + c$, b and c being constants; therefore (m_1, m_2, m_3) are the roots of $am^3 + m(h-2a) + bh + c = 0$ ....(a), where h varies. Take any one of the intersection of tangents $x = am_1m_2$, $y = a(m_1+m_2)$, since m_1, m_2 are two roots of (a), we obtain

$$(h-2a)a = y^2 + ax, \quad abh + ac = xy;$$

whence, eliminating h, we obtain

$$by^2 + abx + 2a^2b = xy - ac, \text{ or } (y-ab)(by-x+ab^2) + a^2b^3 + ac + 2a^2b = 0,$$

an hyperbola with asymptotes $y = ab$ and $x = b(y+ab)$.

7275. (By D. EDWARDES.)—If $\tan\alpha\cot\frac{1}{2}(\beta+\gamma) = \tan\beta\cot\frac{1}{2}(\gamma+\alpha)$, prove that $\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha) = 0$.

Solution by R. W. WHITE, B.A.; R. KNOWLES, B.A., L.C.P.; *and others.*

Reducing successively, we have

$$\frac{\sin\alpha\cos\frac{1}{2}(\beta+\gamma)}{\cos\alpha\sin\frac{1}{2}(\beta+\gamma)} = \frac{\sin\beta\cos\frac{1}{2}(\gamma+\alpha)}{\cos\beta\sin\frac{1}{2}(\gamma+\alpha)},$$

$$\frac{\sin[\alpha+\frac{1}{2}(\beta+\gamma)]}{\sin[\alpha-\frac{1}{2}(\beta+\gamma)]} = \frac{\sin[\beta+\frac{1}{2}(\alpha+\gamma)]}{\sin[\beta-\frac{1}{2}(\alpha+\gamma)]},$$

$$\cos\tfrac{1}{2}(3\alpha-\beta+2\gamma)-\cos\tfrac{1}{2}(\alpha-3\beta-2\gamma)-\cos\tfrac{1}{2}(\alpha+3\beta)+\cos\tfrac{1}{2}(3\alpha+\beta) = 0,$$

$$\sin(\beta-\alpha)\sin[\tfrac{1}{2}(\alpha+\beta)+\gamma]+\sin(\alpha+\beta)\sin\tfrac{1}{2}(\beta-\alpha) = 0,$$

$$2\cos\tfrac{1}{2}(\beta-\alpha)\sin[\tfrac{1}{2}(\alpha+\beta)+\gamma]+\sin(\alpha+\beta) = 0,$$

$$\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta) = 0.$$

7425. (By Professor WOLSTENHOLME, M.A., D.Sc.)—If ABCD be a tetrahedron in which $AB+AC = DB+DC$, prove that $\widehat{AB}+\widehat{AC} = \widehat{DB}+DC$, where $\widehat{AB}$ is the dihedral angle between the planes meeting in AB.

Solution by G. HEPPEL, M.A.; SARAH MARKS; *and others.*

Let the angles $\widehat{AB}, \widehat{AC}, \widehat{DB}, \widehat{DC} = \alpha, \beta, \gamma, \delta$; the edges AB, AC, DB, DC $= a, b, c, d$; $\widehat{AD} = \theta$, AD $= p$, BC $= q$; then,

$$\sin\alpha = \sin\theta\frac{\sin CAD}{\sin BAC}, \quad \cos\beta = \frac{\cos BAD-\cos BAC\cos CAD}{\sin BAC\sin CAD},$$

therefore $\sin\alpha\cos\beta = \dfrac{\sin\theta}{\sin^2 BAC}[\cos BAD-\cos BAC\cos CAD]$.

Similarly, $\sin\beta\cos\alpha = \dfrac{\sin\theta}{\sin^2 BAC}[\cos CAD-\cos BAC\cos BAD]$,

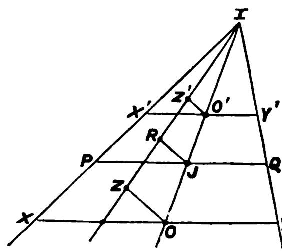
whence $\qquad \sin(\alpha+\beta) = \dfrac{\sin\theta(\cos BAD+\cos CAD)}{1+\cos BAC}$,

therefore $\qquad \dfrac{\sin(\alpha+\beta)}{\sin\theta} = \dfrac{(a+b)p^2+(a^2-c^2)b+(b^2-d^2)a}{(a+b)^2-q^2}$.

By interchanging a and b with c and d, we obtain an expression for $\dfrac{\sin(\gamma+\delta)}{\sin\theta}$; and, if $a+b = c+d$, the denominators and first terms of the numerators are the same in each, and the difference of the remaining terms is $(a^2-c^2)(b+d)+(b^2-d^2)(a+c)$ or $(a+c)(b+d)(a+b-c-d)$; so that, if $a+b = c+d$, $\alpha+\beta = \gamma+\delta$.

7490. (By Professor WOLSTENHOLME, M.A., Sc.D.)—At each point of a central conic is described the rectangular hyberbola of closest contact; prove that the locus of its centre is the inverse of the conic with respect to the director-circle.

Solution by Professor TOWNSEND, F.R.S.



This pretty result is manifestly a particular case of the following more general property—that, if O and O′ (see figure) be the centres of any two conics U and U′ having double contact at any two points P and Q, I the intersection of their common tangents at those points, and J the middle point of their chord of contact PQ, the two points I and J being of course collinear with O and O′; then, when director-circle of U′ is evanescent, rectangle $OI - OO'$ = radius² of director-circle of U. Which may be readily proved as follows:—Since, manifestly, XY and X′Y′ being the parallels through O and O′ to PQ,

$$\text{radius}^2 \text{ of director-circle of } U = OI - OJ + OY - JQ,$$

$$\text{radius}^2 \text{ of director-circle of } U' = O'I - O'J + O'Y' - JQ,$$

therefore when the latter circle is evanescent, and when consequently by similar triangles $OI - O'J + OY - JQ = 0$, then, by subtraction, radius² of director-circle of $U = OI\,(OJ - O'J) = OI - OO'$, and therefore, &c.

The analogous property in Geometry of three dimensions—viz., that, of all central quadrics having closest contact with a given central quadric at any point of its surface, the centre of that whose director-sphere is evanescent is the inverse of the point of contact with respect to the director-sphere of the given quadric—is manifestly also a particular case of the more general property that, if O and O′ (same figure) be the centres of any two quadrics U and U′ having triple contact at any three points P, Q, R, I the intersection of their common tangent-planes at those points, and J the centre of their conic of contact PQR, the two points I and J being of course again collinear with O and O′; then, when director-sphere of U′ is evanescent, rectangle $OI - OO'$ = radius² of director-sphere of U. Which again may be proved nearly similarly with its analogue as follows:—Since manifestly, for any pair of conjugate radii JQ and JR of the conic of contact, XYZ and X′Y′Z′ being the parallels through O and O′ to its plane PQR,

$$\text{radius}^2 \text{ of director-sphere of } U = OI - OJ + OY - JQ + OZ - JR,$$

$$\text{radius}^2 \text{ of director-sphere of } U' = O'I - O'J + O'Y' - JQ + O'Z' - JR.$$

Therefore, when the latter sphere is evanescent, and when consequently by similar triangles $OI - O'J + OY - JQ + OZ - JR = 0$; then, by subtraction, radius² of director-sphere of $U = OI\,(OJ - O'J) = OI - OO''$, and therefore, &c.

The conic or quadric U′, in the above general property corresponding to the case, with the line or plane of contact PQ or PQR, being supposed to remain fixed, and U on the contrary to vary, and its centre O to assume in consequence every possible position on the fixed line IJ; it follows at once, from the above general equation corresponding to the case, viz., (radius)² of director circle or sphere of $U = OO' - OI$, that

the system of director-circles or spheres determined by the variation of U has a common radical axis or plane, situated in each case midway between the two points O' and L which are in each case the two limiting points of the system,—a particular case, manifestly, of the still more general but probably better known property corresponding to the case, that, for every system of conics having four common tangent lines, or of quadrics having eight common tangent planes, the director circles or spheres determine a coaxal or coplanar system, the two limiting points of which are the centres of the two conics or quadrics of the system whose director circles or spheres are evanescent.

7465. (By G. Heppel, M.A.)—In a recent Cambridge Higher Local Examination, the following question was set:—"If n be a prime number, prove that $(x+y)^n - x^n - y^n$ is divisible by $nxy(x+y)(x^2+xy+y^2)$." This being assumed, determine the general term of the quotient.

Solution by the Proposer.

Let P be the expression $(x+y)^n - x^n - y^n$; then

$$\frac{P}{nxy} = x^{n-2} + \frac{n-1}{2\,!}\,x^{n-3}y + \frac{(n-1)(n-2)}{3\,!}\,x^{n-4}y^2 + \&c.$$

Now, let $A_1 A_2 A_3$, &c., $B_1 B_2 B_3$, &c., $C_1 C_2 C_3$, &c., be the respective coefficients of

$(x-y)\,P + nxy,\quad (x-y)\,P + nxy\,(x^3-y^3),\quad (x-y)\,P + nxy\,(x^3-y^3)(x+y)$;

then $C_1 C_2 C_3$, &c. are the coefficients we require to know.

The following laws are evident from either multiplication or division

$$A_1 = 1,\ A_2 = \frac{n-3}{2\,!},\ A_3 = \frac{(n-1)(n-5)}{3\,!},\ A_4 = \frac{(n-1)(n-2)(n-7)}{4\,!},\ \&c.;$$

$B_1 = A_1,\ B_2 = A_2,\ B = A_3,\ B_4 - B_1 = A_4,\ B_5 - B_2 = A_5,\ B_6 - B_3 = A_6,$
and thence we obtain the general relation

$$B_t = A_t + A_{t-3} + A_{t-6} + A_{t-9} + \&c. \ldots$$

Now C_t and the B coefficients are connected by $B_t = C_t + C_{t-1}$ which necessarily leads to $C_t = B_t - B_{t-1} + B_{t-2} - B_{t-3}$, &c. ..., and this gives as a final relation

$$C_t = (A_t - A_{t-1} + A_{t-2}) + (A_{t-6} - A_{t-7} + A_{t-8}) + (A_{t-12} - A_{t-13} + A_{t-14}) + \&c.$$

As a numerical illustration, let $n = 17$, and let it be required to find the seventh term of the quotient, then

$$C_7 = \frac{16\,.\,15\,.\,14\,.\,13\,.\,12\,.\,4}{7\,!} - \frac{16\,.\,15\,.\,14\,.\,13\,.\,6}{6\,!} + \frac{16\,.\,15\,.\,14\,.\,8}{5\,!} + 1$$

$$= 416 - 364 + 224 + 1 = 277, \text{ so that the seventh term is } 277x^6y^6.$$

With reference to the original question set in the Cambridge Examination, it may be noticed that the necessary and sufficient condition of divisibility is that n must be of the form $6n \pm 1$. This of course includes all primes.

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